Optimality Conditions for Coordinate-Convex Policies in CAC With Nonlinear Feasibility Boundaries

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***Abstract*—Optimality conditions for Call Admission Control (CAC) problems with nonlinearly constrained feasibility regions and K classes of users are derived. The adopted model is a generalized stochastic knapsack, with exponentially distributed interarrival times of the objects. Call admission strategies are restricted to the family of Coordinate-Convex (CC) policies. For $K = 2$ classes of users, both general structural properties of the optimal CC policies and structural properties that depend on the revenue ratio are investigated. Then, the analysis is extended to the case $K > 2$. The theoretical results are exploited to narrow the set of admissible solutions to the associated knapsack problem, i.e., the set of CC policies to which an optimal one belongs. With respect to results available in the literature, less restrictive conditions on the optimality of the complete-sharing policy are obtained. To illustrate the role played by the theoretical results on the combinatorial CAC problem, simulation results are presented, which show how the number of candidate optimal CC policies dramatically decreases as the derived optimality conditions are imposed.**

***Index Terms*—Call Admission Control (CAC), combinatorial optimization, Coordinate-Convex (CC) policies, feasibility region, nonlinear constraints, Stochastic Knapsack problem.**

I. Introduction

CALL Admission Control (CAC) determines when to accept or reject a new connection, flow, or call request, thus limiting the load that enters a network. This is accomplished by verifying if enough resources are available to satisfy the performance requirements (in terms, e.g., of packet loss, delay, and jitter) of an incoming call without penalizing those already in progress, in such a way to maximize an objective represented, e.g., by the expected revenue associated with the accepted calls. Therefore, CAC can be exploited to guarantee specific *quality-of-service* (QoS) requirements on the load entering the network.

Manuscript received February 03, 2012; revised July 30, 2012 and September 27, 2012; accepted September 29, 2012; approved by IEEE/ACM TRANSACTIONS ON NETWORKING Editor E. Modiano. Date of publication November 16, 2012; date of current version October 11, 2013. An abridged version of some of the results presented in Section III-C appeared in the Proceedings of the IEEE INFOCOM Mini-Conference 2011.

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Digital Object Identifier 10.1109/TNET.2012.2222924

A basic model for CAC is the combinatorial optimization problem known as *Stochastic Knapsack* [2] (see [3, Ch. 2–4] for an in-depth exposition), in which one has C resource units and $K \geq 2$ classes of users. The calls from each class $k \in \mathcal{K} := \{1, \ldots, K\}$ arrive with exponentially distributed interarrival times (e.g., according to a Poisson process). If accepted by the system, each of them occupies b_k resource units (e.g., bandwidth), which are released at the end of the call. The simplest CAC policy, known as *Complete Sharing* (CS), consists of accepting a call whenever the system has sufficient resource units. However, CS may lead to a monopolistic use of resources by certain classes of users [4, Sec. III]. This motivates the interest in other admission policies [5, Sec. 7.1].

In general, finding optimal policies for the stochastic knapsack is a difficult combinatorial optimization problem [3, Ch. 4]. The *a priori* knowledge of structural properties of the optimal policies is useful to narrow their search. For instance, for two classes of users and an objective given by a weighted sum of per-class average revenues, structural properties were derived in [2] for the optimal *Coordinate-Convex policies* (CC policies), whose definition is recalled in Section II. In practice, such properties restrict the call state $(n_1, \ldots, n_K)$ of the CAC system associated with the stochastic knapsack to suitable subsets of $\{(n_1, \ldots, n_K) \in \mathbb{N}_0^K : \sum_{k \in \mathcal{K}} n_k b_k \leq C\}$, where each n_k represents the number of calls of the kth class accepted by the system and currently in progress. CC policies form a large family of CAC policies, characterized by a relatively simple structure and interesting properties, such as their product-form steady-state distribution [3, Ch. 4] and bounds on per-class blocking probabilities [6]. CC policies and their performance are considered, e.g., in [7]–[10] in various contexts, such as ATM and wireless networks. When service rates and resource requirements do not depend on the customer's classes (*single service*), the optimal CAC policy is not CC and is called *Trunk Reservation* (TR) [11], [12]. Recursive formulas to evaluate the performance of TR were derived in [13] and [14]. However, CC policies cover most practical cases, and they are often taken as a starting point for further analysis. References [13] and [14] propose an iterative algorithm to find a particular kind of CC policy, called *coordinate-optimal threshold policies*, in *multiservice* systems (i.e., systems where different classes may have different and heterogeneous resource requirements and mean service times).

The *feasibility region* [15, pp. 46–49], [16] is a (typically bounded) region Ω_{FR} in the call space, where given QoS requirements in terms, e.g., of packet-loss/packet-delay probability, are statistically guaranteed. In Fig. 1, $(\partial\Omega_{\mathrm{FR}})^+$

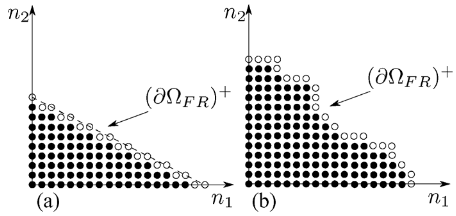

Fig. 1. Upper boundary $(\partial\Omega_{\rm FR})^+$ of a feasibility region $\Omega_{\rm FR}$ with two classes of users in the case of (a) a linearly constrained $\Omega_{\rm FR}$ and (b) a nonlinearly constrained $\Omega_{\rm FR}$.

denotes the (typically nonlinear) upper boundary of $\Omega_{\rm FR}$. There exist various important contexts in which the linear constraint $\{\sum_{k\in\mathcal{K}} n_k b_k \le C\}$ in the stochastic knapsack model, described in Fig. 1(a), has to be replaced by a more complex (not necessarily convex) constraint, which defines a nonlinear feasibility region. This is the case, e.g., with *dynamic service separation* in statistical multiplexing [3], [17]. The underlying idea of dynamic service separation is that only cell streams from the same service (class of users) are allowed to be statistically multiplexed. In an ATM multiplexer, a separate mini-buffer is thus allocated to store cells from each service. The QoS provision for each mini-buffer (i.e. cell loss rate, maximum cell transfer delay, and cell delay variation) can be supported by a weighted round robin or weighted fair queueing scheduler and an appropriate assignment of the scheduling weights. For dynamic service separation, the scheduling weight for a given mini-buffer k is made directly proportional to $\beta_k(n_k)$, which denotes the *equivalent capacity* (also known as *capacity function* [3]) associated with the kth mini-buffer. The equivalent capacity $\beta_k(n_k)$ is the minimum amount of link capacity needed in order to meet the packet-level QoS requirements when n_k connections are being served at the kth mini-buffer. To achieve the QoS guarantee for all the mini-buffers that share the same link with total capacity C, the number of ongoing connections n_k must satisfy for all k the capacity requirement constraint $\sum_{k\in\mathcal{K}} \beta_k(n_k) \le C$. Under service separation, it is worth noting that $\beta_k(\cdot)$ depends only on n_k and not on n_j, $j \neq k$. By assigning an *effective bandwidth* b_k^e [3, p. 32] to the kth class, one obtains $\beta_k(n_k) = n_k b_k^e$. However, the concept of effective bandwidth is an approximation of the CAC system. Indeed, it is well known that to reflect the economies of scale in statistically multiplexing cell streams, $\beta_k(n_k)$ has to monotonically increase with decreasing slope as n_k increases [18], [19]. Therefore, in general one has to cope with a nonlinear equivalent capacity, hence a nonlinearly constrained feasibility region.

Another case in which the feasibility region is defined by nonlinear constraints is the *cellular network scenario* (CDMA2000 network) [20]. In [20], the *outage-based admission region* is defined as the region of the call space where the probability of outage (a disconnection of an accepted user) for each user cannot exceed a prespecified threshold. The outage is described by two parameters: the signal-to-interference-plus-noise ratio (SINR) threshold γ and a minimum duration τ. An outage occurs when the SINR remains below the threshold γ for a period longer than or equal to τ. The feasibility region computed in [20] turns out to be nonlinear.

Feasibility regions with convex piecewise-linear constraints arise in *multiservice-multiresource* (MSMR) systems [21]–[25]. In such cases, the constraint $\sum_{k\in\mathcal{K}} n_k b_k \le C$ of the stochastic knapsack is replaced by an N-tuple of similar constraints $\sum_{k\in\mathcal{K}} n_k b_{k,i} \le C_i$, $i = 1,\ldots,N$. These can be further generalized by replacing the terms $n_k b_{k,i}$ with $\beta_{k,i}(n_i)$, $i = 1,\ldots,N$.

A feasibility region with a nonlinear constraint can be also interpreted as being associated to a stochastic knapsack whose size is variable and depends on the number of calls of each class $k = 1,\ldots,K$. As remarked in [3, pp. 139–140], often the nonlinear part of the boundary is difficult to describe, both analytically and by simulations. In such a situation, it is worth investigating structural properties of the optimal policies.

Exploring this issue for the family of CC policies is the aim of this paper, in which we further develop the approach proposed in [1] and [26]. To the best of our knowledge, until now the problem of deriving structural properties of the optimal CC policies in the case of general nonlinearly constrained feasibility regions has received little attention. Some exceptions are [20], [23], [24], [27], and our previous works [1] and [26].

This paper is structured as follows. In Section II, we introduce the CAC problem studied in [2], which we extend to the case of a nonlinearly constrained feasibility region $\Omega_{\rm FR}$. Concerning CAC problems with nonlinearly constrained feasibility regions, CC policies and $K = 2$ classes of users, in Section III we investigate the following:

- general structural properties of the optimal CC policies (Section III-A);
- an algorithm to enumerate all the candidate-optimal CC policies, i.e., those that satisfy the above-mentioned structural properties (Section III-B);
- structural properties that depend on the revenue ratio associated with the two classes of users (Section III-C).

Section IV extends to $K > 2$ the results in Section III. Numerical simulations are reported in Section V. A conclusive discussion and comparisons with the results derived in [20] for the optimality of the complete-sharing policy are contained in Section VI. All the proofs of theorems and propositions are deferred to Section VII.

II. Problem Formulation

The model adopted in [2] for the CAC system is described by a two-dimensional vector $\mathbf{n}$, whose component n_k, $k = 1, 2$, represents the number of connections of class k that have been accepted by the system and are currently in progress. For each class k, the interarrival times are exponentially distributed with mean value $1/\lambda_k(n_k)$ and the holding times of the accepted connections are independent and identically distributed (i.i.d.) with mean value $1/\mu_k$. The CAC system accepts or rejects a connection request according to a CC *policy*, whose definition [3, p. 116] is recalled in Definition II.1.

Definition II.1: A nonempty set $\Omega \subseteq \Omega_{\rm FR} \subsetneq \mathbb{N}_0^2$ is called CC if and only if for each $\mathbf{n} \in \Omega$ with $n_k > 0$, one has $\mathbf{n} - \mathbf{e}_k \in \Omega$, $\forall k = 1, 2$, where $\mathbf{e}_k$ is a two-dimensional vector whose kth component is 1 and the other is 0. The CC policy associated with a CC set Ω admits an arriving request of connection if and only if after admittance the state process remains in Ω.

As there is a one-to-one correspondence between CC sets and CC policies, we use the symbol Ω to denote both a CC set and a CC policy.

We consider the following optimization problem associated with the CAC system:

$$\text{maximize} \quad J(\Omega) = \sum_{\mathbf{n}\in\Omega} (\mathbf{n}\cdot\mathbf{r}) P_\Omega(\mathbf{n}) \tag{1}$$

$$\text{s.t.} \quad \Omega \in \mathcal{P}(\Omega_{\mathrm{FR}}) \tag{2}$$

where $\mathcal{P}(\Omega_{\mathrm{FR}})$ is the set of CC subsets of Ω_{FR}, $\mathbf{r} := (r_1, r_2)$ is a two-dimensional vector whose component r_k represents the instantaneous revenue generated by any accepted connection of class k that is in progress, and $P_\Omega(\mathbf{n})$ is the steady-state probability that the CAC system is in state $\mathbf{n}$. Each r_k may be also considered as a weight to set priorities between the classes. As Ω is CC, $P_\Omega(\mathbf{n})$ is known to take on the product-form expression

$$P_\Omega(\mathbf{n}) = \frac{\prod_{k=1}^{2} q_k(n_k)}{\sum_{\mathbf{m}\in\Omega} \prod_{k=1}^{2} q_k(m_k)} \tag{3}$$

where

$$q_k(n_k) := \frac{\prod_{j=0}^{n_k-1} \lambda_k(j)}{n_k! \mu_k^{n_k}}. \tag{4}$$

In the case of a linearly constrained feasibility region Ω_{FR} ($\{(n_1,\ldots,n_K) \in \mathbb{N}_0^K : \sum_{k\in\mathcal{K}} n_k b_k \leq C\}$), [2] derived sufficient conditions under which the CC policies maximizing the objective (1) are of threshold type (defined in Section III-C). Such conditions depend on the value assumed by the revenue ratio $R := r_2/r_1$.

III. Nonlinear Upper Boundary, Two Classes of Users

In our analysis, we consider the general case in which the feasibility region Ω_{FR} may have a nonlinear upper boundary, denoted by $(\partial\Omega_{\mathrm{FR}})^+$ [see Fig. 1(b)]. We denote by $(\partial\Omega)^+$ the upper boundary of the CC set Ω. The set Ω_{FR} is assumed to be CC, as it often happens for feasibility regions defined in terms of QoS constraints [28, Proposition 6.3]. We recall from [2] the following definition.

Definition III.1: The tuple $(\alpha,\beta) \in \Omega_{\mathrm{FR}} \backslash \Omega$ is a type-1 corner point for Ω if and only if $\beta \geq 1$, $(\alpha, \beta-1) \in \Omega$, and either $\alpha = 0$ or $(\alpha-1,\beta) \in \Omega$. The tuple $(\alpha,\beta) \in \Omega_{\mathrm{FR}} \setminus \Omega$ is a type-2 corner point for Ω if and only if $\alpha \geq 1$, $(\alpha-1,\beta) \in \Omega$, and either $\beta = 0$ or $(\alpha,\beta-1) \in \Omega$.

We use the term "corner point" to refer both to a type-1 and a type-2 corner point.

A. General Structural Properties

Let Ω^{o} denote a generic optimal CC policy (or its associated CC set). Proposition III.2 states that Ω^{o} has a nonempty intersection with the upper boundary $(\partial\Omega_{\mathrm{FR}})^+$ of the feasibility region. Note that this is a nontrivial result, as for any two CC sets $\Omega_1, \Omega_2 \subseteq \Omega_{\mathrm{FR}}$, in general $\Omega_1 \subseteq \Omega_2$ does not imply $J(\Omega_1) \leq J(\Omega_2)$.

Proposition III.2: Ω^{o} has a nonempty intersection with $(\partial\Omega_{\mathrm{FR}})^+$.

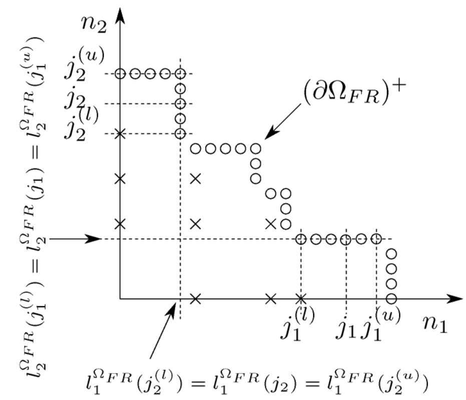

Fig. 2. Potential locations of the corner points of an optimal CC policy Ω^{o}. According to Proposition III.3, the corner points of Ω^{o} have to be searched among the points denoted by crosses.

Given a CC region Ω, we define

$$l_1^\Omega(n_2) := \max\{j_1 \in \mathbb{N}_0 \text{ such that } (j_1,n_2)\in\Omega\} \tag{5}$$

$$l_2^\Omega(n_1) := \max\{j_2 \in \mathbb{N}_0 \text{ such that } (n_1,j_2)\in\Omega\}. \tag{6}$$

The values $l_1^\Omega(n_2)$ and $l_2^\Omega(n_1)$ represent the maximum number of type-1/type-2 connections allowed in Ω when one has already n_2 type-2/n_1 type-1 connections, respectively. It follows from the definitions that the functions $l_k^\Omega(\cdot)$ are nonincreasing. We define $n_{1,\max}^{\mathrm{FR}} := l_1^{\Omega_{\mathrm{FR}}}(0)$, $n_{2,\max}^{\mathrm{FR}} := l_2^{\Omega_{\mathrm{FR}}}(0)$.

Our next Proposition III.3, which extends the property stated in [2, Theorem 1] for the linearly constrained case to nonlinearly constrained feasibility regions, defines all the possible positions of the corner points of an optimal CC policy Ω^{o}. In particular, it states that the corner points of Ω^{o} can be located only among the vertices of a suitable grid[1] denoted by crosses in Fig. 2.

Proposition III.3:

(i) If (α,β) is a type-2 corner point for Ω^{o}, then for some $j_2 = 1,\ldots,n_{2,\max}^{\mathrm{FR}}$ one has

$$\alpha = l_1^{\Omega_{\mathrm{FR}}}(j_2) + 1. \tag{7}$$

(ii) If (α,β) is a type-1 corner point for Ω^{o}, then for some $j_1 = 1,\ldots,n_{1,\max}^{\mathrm{FR}}$ one has

$$\beta = l_2^{\Omega_{\mathrm{FR}}}(j_1) + 1. \tag{8}$$

Let $\mathcal{S}$ be the set of all CC policies whose corner points are among the vertices of the grid G with vertical lines of equations $\alpha = 0$ and $\alpha = l_1^{\Omega_{\mathrm{FR}}}(j_2) + 1$ ($j_2 = 1,\ldots,n_{2,\max}^{\mathrm{FR}}$) and horizontal lines of equations $\beta = 0$ and $\beta = l_2^{\Omega_{\mathrm{FR}}}(j_1) + 1$ ($j_1 = 1,\ldots,n_{1,\max}^{\mathrm{FR}}$). According to Proposition III.3, any optimal CC policy Ω^{o} belongs to $\mathcal{S}$.

For CC policies with at least two corner points, Proposition III.2 can also be obtained as a consequence of the following Proposition III.4, proved in [26]. The proposition implies that between any two successive corner points (ordered increasingly with respect to the coordinate n_1) the

[1]Note, however, that not all the combinations of points in the grid are a feasible choice as corner points. Indeed, due to Definition III.1 and to the coordinate-convexity of Ω, no two corner points can be on the same vertical or horizontal line.

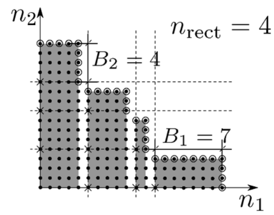

Fig. 3. Decomposition of the feasibility region into discrete disjoint rectangles. The crosses represent the potential locations of the corner points of an optimal CC policy, according to Proposition III.3.

intersection between $(\partial\Omega^{\rm o})^{+}$ and $(\partial\Omega_{\rm FR})^{+}$ is nonempty (see also [26, Fig. 2(b)]).

Proposition III.4 [26]: Let $(\alpha^{(i)}, \beta^{(i)})$ and $(\alpha^{(i+1)}, \beta^{(i+1)})$ be two consecutive corner points of $\Omega^{\rm o}$. Then, the intersection between the vertical line $n_1 = \alpha^{(i+1)} - 1$ and the horizontal line $n_2 = \beta^{(i)} - 1$ either lies on $(\partial\Omega_{\rm FR})^{+}$ or is outside $\Omega_{\rm FR}$.

According to our next proposition, in order to identify a CC policy Ω and, in particular, to find an optimal one $\Omega^{\rm o}$, it is sufficient to search within the set of corner points. More specifically, the proposition states that one can construct a CC policy starting merely from the knowledge of the positions of its corner points.

Proposition III.5: Let $I(\Omega)$ denote the set of corner points $\{(\alpha^{(i)}, \beta^{(i)})\}$ of a CC policy $\Omega \subseteq \Omega_{\rm FR}$ and $C_i^{-} := \{\mathbf{n} \in \Omega_{\rm FR} : n_1 \geq \alpha^{(i)}$ and $n_2 \geq \beta^{(i)}\}$. Then, $\Omega = (\Omega_{\rm FR} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^{-})$.

Proposition III.5 shows that a CC policy Ω can be always obtained by removing particular regions C_i^{-} from the feasibility region $\Omega_{\rm FR}$. The regions C_i^{-} are built by using the corner points of Ω. In the case of an optimal CC policy $\Omega^{\rm o}$, the search is simplified by the fact that its corner points belong to the grid G, so $I(\Omega^{\rm o}) \subseteq G$. An interesting result related to Proposition III.5 was stated in [23, Theorem 2] (in the general case $K \geq 2$), where it was proved that for feasibility regions corresponding to MSMR systems, the optimal CC policies $\Omega^{\rm o}$ are obtained by removing from the feasibility region $\Omega_{\rm FR}$ all the points that belong to the intersection of a finite number of hyperplanes. In general, such a number of hyperplanes is not known *a priori*, but by further restricting the search for the optimal policies to *convex* CC subsets of $\Omega_{\rm FR}$, stronger optimality conditions were provided [23, Corollary 1]. Moreover, [24] shows that any optimal CC policy is convex for a continuous relaxation of the problem considered in [23].

B. Cardinality of the Set of Candidate Optimal CC Policies

The feasibility region can always be written as the union of a finite number $n_{\rm rect}$ of discrete disjoint rectangles with bases on the n_1-axis or the n_2-axis, as shown in Fig. 3. Note from Fig. 3 that the cardinality $|G|$ of the grid G is

$$\left(\sum_{i=1}^{n_{\rm rect}} i\right) - 1 = \frac{n_{\rm rect}(n_{\rm rect}+1)}{2} - 1$$

so it is a function of $n_{\rm rect}$ only. Similarly, the cardinality $|\mathcal{S}|$ of the set $\mathcal{S}$ is a function of $n_{\rm rect}$ only; let us denote such a cardinality by $|\mathcal{S}(n_{\rm rect})|$. In general $|\mathcal{S}(n_{\rm rect})|$ is much smaller than the cardinality $2^{|G|}$ of the power set of G. Indeed, if a candidate optimal CC policy has a corner point that is a vertex of G, then some other vertices of G cannot be corner points for that policy. More precisely, for two successive corner points $(\alpha^{(i)}, \beta^{(i)})$ and $(\alpha^{(i+1)}, \beta^{(i+1)})$ with $\alpha^{(i)} < \alpha^{(i+1)}$, the coordinate-convexity of the policy imposes the constraint $\beta^{(i)} > \beta^{(i+1)}$. In particular, this implies that any optimal CC policy has at most

$$n_{\rm rect} \leq \min\left\{n_{1,\max}^{\rm FR}, n_{2,\max}^{\rm FR}\right\}$$

corner points.

Algorithm 1 provides a way to compute $|\mathcal{S}(n_{\rm rect})|$ by using the following conditions (s_i being the ith element of the currently examined set S, and s an element of S): a) all the coordinates of s are smaller than or equal to the corresponding coordinates of s_i; b) all the coordinates of s are greater than or equal to the corresponding coordinates of s_i.

The correctness of Algorithm 1 can be verified as follows.

Algorithm 1: Enumeration of the CC policies with all corner points on the grid G

Data: A subset S of potential corner points on the grid G. s_i denotes the ith element of S (the ordering is chosen arbitrarily).

1 ***function*** $\mathrm{nPolicies}(S)$
2 {**if** $S = \emptyset$ **then**
3 $\quad \mathrm{nPolicies} \leftarrow 1$
4 **else**
5 $\quad \mathrm{nPolicies} \leftarrow 0$
6 $\quad$ **for** $(i = 1, \ldots, |S|)$ **do**
7 $\qquad S_{\leq i} \leftarrow \{s_j \in S$ with $j \leq i\}$
8 $\qquad R \leftarrow \{s \in S$ meeting cond. a) or b) w.r.t. $s_i\}$
9 $\qquad H \leftarrow S \setminus (S_{\leq i} \cup R)$
10 $\qquad \mathrm{nPolicies} \leftarrow \mathrm{nPolicies} + \mathrm{nPolicies}(H)$
11 $\quad$ **end**
12 $\quad \mathrm{nPolicies} \leftarrow \mathrm{nPolicies} + 1$
13 **end**
14 **return** nPolicies}
15 The number of all CC policies with corner points on the grid G is then computed as $\mathrm{nPolicies}(G)$.

Algorithm 1 constructs each CC policy Ω with all its corner points on the grid G by adding corner points one after the other. Each time a point s_i is inserted into the current set[2] of corner points (line 6 in Algorithm 1), all the points s satisfying conditions a) or b) with respect to s_i are removed from the set of potential corner points of Ω (line 9). Indeed, they cannot be corner points of Ω since this is in contrast with the coordinate-convexity of Ω and the fact that s_i has already been chosen as a corner point of Ω. Moreover, according to line 7, all the CC policies generated by Algorithm 1 have different sets of corner points, so each CC policy is counted exactly once by the algorithm. In particular, the number of policies is incremented by one unit in two cases: either when the current set S is empty (line 3), or when one has decided to exclude all the points in S from being corner points of the current Ω (line 12). In both

[2]One can check that by construction s_i is a corner point, in the sense that it satisfies Definition III.1 and is compatible with all the other corner points that have been added so far in the previous calls of the recursive function "nPolicies" of Algorithm 1.

cases, the set of all the corner points of the current Ω can be obtained by a backward examination of the choices of the index i made in subsequent nested calls of the function "nPolicies" at line 10.

As an example, referring to the feasibility region in Fig. 3, Algorithm 1 provides $|\mathcal{S}(n_{\rm rect})| = 1, 4, 13, 41$ for $n_{\rm rect} = 1, 2, 3, 4$, respectively. Modifications of Algorithm 1 allow one to compute also the exact number of all CC subsets of $\Omega_{\rm FR}$ (i.e., the number of all its CC policies) or lower and upper bounds on such a number. The number of candidate optimal CC policies can be further reduced by imposing compatibility with the necessary optimality conditions stated in Propositions III.2 and III.4 (see also Section V for an example). Both numbers can be computed by suitable modifications of Algorithm 1, not detailed here due to space constraints.

C. Structural Properties Dependent on the Revenue Ratio

Let us now consider structural properties of the optimal CC policies obtained for suitable values of the *revenue ratio* $R := r_2/r_1$. The following definitions are needed.

Definition III.6 [2]: For $a, b \in \mathbb{N}_0$, $a \leq b$, we define

$$x_k(a,b) := \frac{\sum_{j=a}^{b} j q_k(j)}{\sum_{j=a}^{b} q_k(j)} \tag{9}$$

where $q_k(j)$ is defined in (4). $x_k(a,b)$ represents the expected value of the random variable $X_k(a,b)$ corresponding to the equilibrium state of a birth–death process with birth rates $\bar{\lambda}_k(j)$ (for $j \geq 0$) given by

$$\bar{\lambda}_k(j) = \begin{cases} \lambda_k(j), & j < b \\ 0, & j = b \end{cases}$$

and death rates $\bar{\mu}_k(j)$ (for $j \geq 1$) given by

$$\bar{\mu}_k(j) = \begin{cases} j\mu_k, & j > a \\ 0, & j = a. \end{cases}$$

Definition III.7 [2]: Let $k = 1, 2$. A CC policy Ω is threshold type-k if and only if for some $t_k = 0, \ldots, n_{k,\max}^{\rm FR}$ one has

$$\Omega = \{(n_1, n_2) \in \Omega_{\rm FR} : n_k \leq t_k\}.$$

The following theorem states that under suitable conditions, one has threshold-type optimal CC policies. The main idea of the proof consists in deriving conditions that a point of the grid G has to satisfy in order to be a corner point for $\Omega^{\rm o}$ (see Lemma VII.6). When such conditions are not satisfied for a suitable subset of the grid, then $\Omega^{\rm o}$ is of threshold type (see Lemma VII.7). The result is an extension of [2, Theorem 1] to feasibility regions with a nonlinear upper boundary.

Definition III.8: Let B_1 and B_2 be the maximum width and the maximum height of a step in the upper boundary of the feasibility region $(\partial\Omega_{\rm FR})^+$, respectively.

Refer to Fig. 3 for an example of B_1 and B_2.

Theorem III.9: Let $\lambda_k(\cdot)$ be nonincreasing for $k = 1, 2$ and $R := r_2/r_1$.

(i) If $R > x_1(0, B_1)$, then $\Omega^{\rm o}$ is threshold type-1 and the threshold is equal to $l_1^{\Omega_{\rm FR}}(j_2)$ for some $j_2 = 0, \ldots, n_{2,\max}^{\rm FR}$.

(ii) If $(1/R) > x_2(0, B_2)$, then $\Omega^{\rm o}$ is threshold type-2 and the threshold is equal to $l_2^{\Omega_{\rm FR}}(j_1)$ for some $j_1 = 0, \ldots, n_{1,\max}^{\rm FR}$.

(iii) If $x_1(0, B_1) < R < (1/x_2(0, B_2))$, then $\Omega^{\rm o} = \Omega_{\rm FR}$.

An extension of [2, Theorem 1] to a nonlinearly constrained feasibility region less general than ours and under a different assumption on the holding-time distribution of the calls was reported in [27, Sec. 4].

IV. Extension to $K \geq 2$ Classes

Let the number of classes $K \geq 2$ and, for a nonlinearly constrained feasibility region, consider the problem formulation given in Section II with all the two-dimensional vectors replaced by correspondent K-dimensional ones.

Definition IV.1: A nonempty set $\Omega \subseteq \Omega_{\rm FR} \subsetneq \mathbb{N}_0^K$ is called CC if and only if for every $\mathbf{n} \in \Omega$ with $n_k > 0$ one has $\mathbf{n} - \mathbf{e}_k \in \Omega$, where $\mathbf{e}_k$ is a K-dimensional vector whose kth component is 1 and the others are 0. The CC policy associated with a CC set Ω admits an arriving request of connection if and only if after admittance the state process remains in Ω.

Definition IV.1: Given a K-dimensional Cartesian coordinate system, for $k \in \mathcal{K}$ and $j \in \mathbb{N}_0$, let

$$\mathcal{P}_k(j) := \{\mathbf{n} \in \mathbb{N}_0^K : n_k = j\}$$

be the $\mathcal{P}_k(j)$ the $(K-1)$-dimensional discrete hyperplane at the jth position along the kth axis.

Definition IV.3: For $S \subsetneq \mathbb{N}_0^K$ and $k = 1, \ldots, K$, let $\pi_{\mathcal{K}\setminus\{k\}}(S)$ be the projection of S on $\mathcal{P}_k(0)$, i.e., along the kth axis. For a feasibility region $\Omega_{\rm FR} \subsetneq \mathbb{N}_0^K$ and $k = 1, \ldots, K$, let $n_{k,\max}^{\rm FR}$ be the largest index j such that $\pi_{\mathcal{K}\setminus\{k\}}(\Omega_{\rm FR} \cap \mathcal{P}_k(j)) \neq \emptyset$.

Definition IV.4: For a feasibility region $\Omega_{\rm FR} \subsetneq \mathbb{N}_0^K$, the grid $G \subseteq \Omega_{\rm FR}$ is defined as $G := \{\cap_{k=1,\ldots,K}\mathcal{P}_k : \mathcal{P}_k \in \mathcal{A}_k\} \cap (\Omega_{\rm FR} \setminus \{0, 0, \ldots, 0\})$, where the sets $\mathcal{A}_k$ are constructed as described in Algorithm 2.

Algorithm 2: Construction of the grid G

Data: We take K orthogonal $(K-1)$-dimensional discrete hyperplanes $\mathcal{P}_k(0)$, $k = 1, \ldots, K$.
1 **foreach** $k = 1, \ldots, K$ **do**
2 $\quad \mathcal{A}_k \leftarrow \mathcal{P}_k(0)$
3 $\quad\quad \mathcal{B}_k \leftarrow \pi_{\mathcal{K}\setminus\{k\}}(\Omega_{\rm FR} \cap \mathcal{P}_k(0))$
4 $\quad$ **for** $(j = 1, \ldots, n_{k,\max}^{\rm FR})$ **do**
5 $\quad\quad$ **if** $\pi_{\mathcal{K}\setminus\{k\}}(\Omega_{\rm FR} \cap \mathcal{P}_k(j)) \subsetneq \mathcal{B}_k$ **then**
6 $\quad\quad\quad \mathcal{A}_k \leftarrow \mathcal{A}_k \cup \mathcal{P}_k(j)$
7 $\quad\quad\quad \mathcal{B}_k \leftarrow \pi_{\mathcal{K}\setminus\{k\}}(\Omega_{\rm FR} \cap \mathcal{P}_k(j))$
8 $\quad\quad$ **end**
9 $\quad$ **end**
10 **end**
11 The points of G are obtained as intersections of hyperplanes in $\mathcal{A}_1, \ldots, \mathcal{A}_K$ and $\Omega_{\rm FR}$, excluding the point (0, 0, …, 0) that does not belong to the grid.

The following is an informal description of Algorithm 2. Each hyperplane $\mathcal{P}_k(j)$ (line 1 in Algorithm 2) moves along the kth axis starting from $j = 0$ and scanning the whole feasibility region $\Omega_{\rm FR}$ (line 4). When the cross section

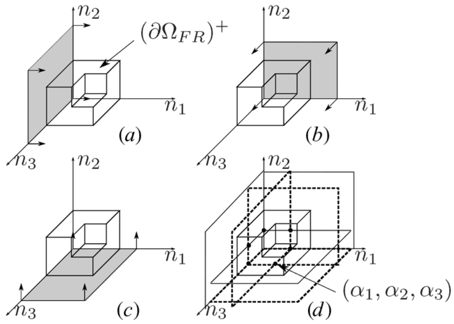

Fig. 4. Illustration of the procedure presented in Definition IV.4, Algorithm 2. The points of the grid are obtained as intersections, as shown in (d), of suitable planes shown in (a)–(c) for the point $(\alpha_1, \alpha_2, \alpha_3)$.

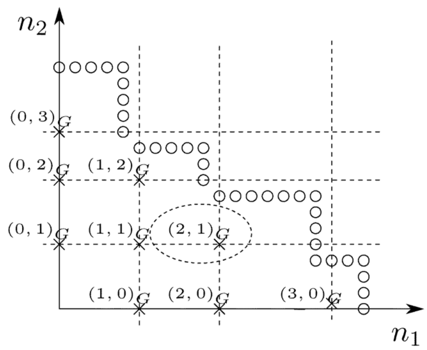

Fig. 5. Example of enumeration of the points in the grid G, with $K = 2$.

$\pi_{\mathcal{K}\setminus\{k\}}(\Omega_{\rm FR} \cap \mathcal{P}_k(j))$ changes (line 5), the position j of the hyperplane is recorded in $\mathcal{A}_k$ (line 6) together with the current cross section (line 7). The hyperplane with $j = 0$ is always considered (line 2). The intersections among the feasibility region and the recorded discrete hyperplanes in the sets $\mathcal{A}_k$ form the points that compose the grid G (line 11). By definition, the point (0, 0, …, 0) does not belong to the grid.

Fig. 4 shows a feasibility region $\Omega_{\rm FR} \subsetneq \mathbb{N}_0^3$; for the sake of simplicity, only $(\partial\Omega_{\rm FR})^+$ is drawn and represented as a continuous contour. Analogously, the discrete planes $\mathcal{P}_k$ are represented as continuous planes.

To identify a specific point $\mathbf{g} \in G$, we use the following notation. Given a grid G in a K-dimensional region, we associate each point $\mathbf{g}$ in G with a $1 \times K$ vector $\mathbf{C}_G$, whose kth component represents the position of the point along the kth axis. We denote by $\mathbf{C}_{G,k}$ the kth component g_k of the point $\mathbf{g}$ associated with $\mathbf{C}_{G,k}$. Refer to Fig. 5 for a graphical explanation in which the point $(2,1)_G$ in a two-dimensional region is highlighted.

Definition IV.5: Given $\mathbf{a}, \mathbf{b} \in G \subsetneq \mathbb{N}_0^K$, $\mathbf{b}$ is consecutive to $\mathbf{a}$ with respect to $k \in \mathcal{K}$ if and only if

$$\begin{cases} a_j = b_j & \forall j \in \mathcal{K} \setminus \{k\} \\ b_k > a_k & \\ \nexists \mathbf{c} \in G : c_j = a_j = b_j & \forall j \in \mathcal{K} \setminus \{k\}; c_k \in (a_k, b_k). \end{cases}$$

Of course, Definition IV.5 extends to the two vectors associated with the two points $\mathbf{a}, \mathbf{b} \in G$. Given two vectors $\mathbf{C}_G$, $(\mathbf{C} + \mathbf{e}_k)_G$ associated with two points in the grid G, with $\mathbf{e}_k$ a $1 \times K$ vector whose kth component is 1 and the other ones are 0, it follows from Definition IV.5 that $(\mathbf{C} + \mathbf{e}_k)_G$ is consecutive to $\mathbf{C}_G$ with respect to k (see Fig. 5). The following is an extension of Definition III.1 to the multidimensional case. In the two-dimensional case, in order to simplify the notation in the previous sections, we denoted the second index α_2 by β.

Definition IV.6: Given a CC policy Ω, the K-tuple $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_K) \in \Omega_{\rm FR} \setminus \Omega$ is a corner point for Ω if and only if $\forall k \in \mathcal{K}$ such that $\alpha_k \neq 0$, one has $\boldsymbol{\alpha} - \mathbf{e}_k \in \Omega$.

The following two results are extensions to the K-dimensional case of Proposition III.3 and Theorem III.9, respectively.

Proposition IV. 7: If the K-tuple $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_K)$ is a corner point of an optimal CC policy $\Omega^{\rm o}$, then it belongs to the grid G.

Theorem IV.8: Let $\lambda_k(\cdot)$ be nonincreasing for $k = 1, \ldots, K$. The following holds.

(i) If for a given $k \in \mathcal{K}$ and for all the vectors $\mathbf{C}_G$ with $\mathbf{C}_{G,k} \neq 0$, one has

$$r_k > \frac{\sum_{j \in \mathcal{K}\setminus\{k\}} r_j x_j \left(0, n_{j,\max}^{\rm FR}\right)}{\mathbf{C}_{G,k} - x_k\left((\mathbf{C} - \mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1\right)} \tag{10}$$

then all the possible corner points of $\Omega^{\rm o}$ have $\alpha_k = 0$.

(ii) If condition (10) holds for all $k \in \mathcal{K}$ and for all the vectors $\mathbf{C}_G$ with $\mathbf{C}_{G,k} \neq 0$, then $\Omega^{\rm o} = \Omega_{\rm FR}$.

Extending the terminology used for 2 classes of users, we call *threshold type-k policy* any CC policy Ω of the form $\Omega = \{(n_1, \ldots, n_K) \in \Omega_{\rm FR} : n_k \leq t_k\}$ for some nonnegative integer t_k. All the corner points of such a policy have their hth coordinate $\alpha_h = 0$, for $h \neq k$. When the assumptions of Theorem IV.8(i) hold for all $h \neq k$, it follows that $\Omega^{\rm o}$ is a threshold type-k policy, and by Proposition IV.7 its unique corner point also belongs to the grid G.

Finally, we mention that Propositions III.2 and III.5 hold also for the case[3] $K > 2$; due to space limits, their proofs are only sketched at the end of Section VII. Moreover, for $K > 2$ Algorithm 1 can be used (with no variation) to enumerate all the CC policies with corner points on the grid G.

V. Simulation Results

A. Number of CC Policies Satisfying Various Constraints

In this section, we show how the number of candidate optimal CC policies dramatically decreases as the necessary optimality conditions that we have derived are added. Four different scenarios are simulated. They are enumerated from 1 to 4 and differ for the following:

- the number of classes and the relative feasibility region: $K = 2$ and $\Omega_{\rm FR}^{(1)}$ for Scenarios 1 and 2 and $K = 3$ and $\Omega_{\rm FR}^{(2)}$ for Scenarios 3 and 4;
- the shape of the optimal CC policy $\Omega^{\rm o}$: in Scenario 2 (4, respectively), the assumptions of Theorem III.9(i) [Theorem IV.8(i) for $k = 2$ and 3, respectively] are satisfied, and any optimal CC policy $\Omega^{\rm o}$ is a threshold type-1, whereas in Scenarios 1 and 3, none of the conditions stated in Theorem III.9(i) [Theorem IV.8(ii), respectively] is satisfied, so the shape of $\Omega^{\rm o}$ is not provided by such theorems.

Table I summarizes the scenarios used in the simulations. To determine the two feasibility regions $\Omega_{\rm FR}^{(1)} \subsetneq \mathbb{N}_0^2$ and $\Omega_{\rm FR}^{(2)} \subsetneq$

[3]For $K > 2$, the sets C_i^- are defined as $C_i^- := \{\mathbf{n} \in \Omega_{\rm FR} : n_k \geq \alpha_k^{(i)},$ for $k = 1, \ldots, K\}$, i.e., they are K-dimensional hyperoctants.

TABLE I
SCENARIOS SIMULATED

Name	Users' classes	Feasibility Region	Optimal CC policy
Scenario 1	$K = 2$	$\Omega_{FR}^{(1)}$	Ω^o threshold type-1
Scenario 2	$K = 2$	$\Omega_{FR}^{(1)}$	Ω^o of unknown shape
Scenario 3	$K = 3$	$\Omega_{FR}^{(2)}$	Ω^o threshold type-1
Scenario 4	$K = 3$	$\Omega_{FR}^{(2)}$	Ω^o of unknown shape

TABLE II
TRAFFIC AND PACKET-LEVEL QoS PARAMETERS USED TO DETERMINE $\Omega_{\mathrm{FR}}^{(1)}$ AND $\Omega_{\mathrm{FR}}^{(2)}$

	$\mathbf{K=2,\Omega_{FR}^{(1)}}$	$\mathbf{K=3,\Omega_{FR}^{(2)}}$
Traffic source model	ON-OFF model	
Server link	250Mbit/s	90Mbit/s
Peak rate	[8Mbit/s; 20Mbit/s]	[20Mbit/s; 20Mbit/s; 12Mbit/s]
ON period	[1s; 10s]	[15s; 15s; 1s]
Utilization	[0.3; 0.3]	[0.8; 0.8; 0.6]
Mini-buffers size	5Mbit	5Mbit
Epsilon	[0.1; 0.001]	[0.003; 0.003; 0.08]

TABLE III
CONNECTION-LEVEL TRAFFIC PARAMETERS

Name	Traffic parameters	Revenues
Scenario 1	$\lambda_1 = 50$; $\lambda_2 = 150$ $\mu_1 = 0.5$; $\mu_2 = 5$	$r_1 = 0.25$; $r_2 = 2.5$ $R = 10$
Scenario 2	$\lambda_1 = 50$; $\lambda_2 = 150$ $\mu_1 = 0.5$; $\mu_2 = 5$	$r_1 = 1.75$; $r_2 = 2.5$ $R = 7$
Scenario 3	$\lambda_1 = 50, \lambda_2 = 30, \lambda_3 = 20$ $\mu_1 = 70, \mu_2 = 50, \mu_3 = 45$	$r_1 = 4, r_2 = 6, r_3 = 7$
Scenario 4	$\lambda_1 = 50, \lambda_2 = 30, \lambda_3 = 20$ $\mu_1 = 70, \mu_2 = 50, \mu_3 = 45$	$r_1 = 4, r_2 = 5, r_3 = 5$

$\mathbb{N}_0^3$, we consider the following model. A network node is modeled as a statistical multiplexer with *service separation with dynamic partition*, in which the feasibility region is characterized by the constraint $\sum_{k\in\mathcal{K}} \beta_k(n_k) \leq C$. Assuming ON–OFF traffic sources, we use the results from [18] to compute the following approximation of the capacity function $\beta_k(n_k)$:

$$\beta_k(n_k) \simeq \min\{D : p(n_k, D, S) < \epsilon_k\} \tag{11}$$

where $p(n_k, D, S)$ is the packet loss probability associated with n_k sources of class k sharing a bandwidth D and a mini-buffer of fixed size S. Then, $\beta_k(n_k)$ is approximated as the minimum value of bandwidth D in order to maintain the packet loss probability of packets belonging to the n_k accepted connections below a determinate value $\epsilon_k > 0$, for a fixed S.

Table II shows the parameters used to compute $\Omega_{\mathrm{FR}}^{(1)}$ and $\Omega_{\mathrm{FR}}^{(2)}$. The feasibility regions $\Omega_{\mathrm{FR}}^{(1)}$ and $\Omega_{\mathrm{FR}}^{(2)}$ are shown in Figs. 6 and 7, respectively.

Table III shows the connection-level traffic parameters, structured into four scenarios, used in the simulations. In particular, in Scenario 1, with the feasibility region $\Omega_{\mathrm{FR}}^{(1)}$, we have $B_1 = 8$ and $B_2 = 1$; for $\lambda_1 = 50$, $\lambda_2 = 150$, $\mu_1 = 0.5$, $\mu_2 = 5$, $r_1 = 0.25$, and $r_2 = 2.5$, we have $R = r_2/r_1 = 10$, $x_1(0,8) \simeq 7.92$ and $x_2(0,1) \simeq 0.97$. Then, $R > x_1(0,8)$ and by Theorem III.9(i), there exists an optimal CC policy that is threshold type-1. An upper bound on the number of such threshold type-1 policies is simply $n_{1,\max}^{\mathrm{FR}} + 1$. One can obtain a smaller upper bound by observing that the location

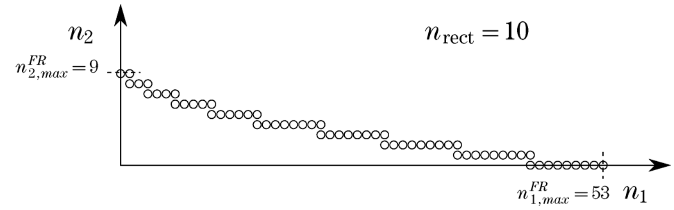

Fig. 6. Feasibility region $\Omega_{\mathrm{FR}}^{(1)}$ determined by the parameters in Table II.

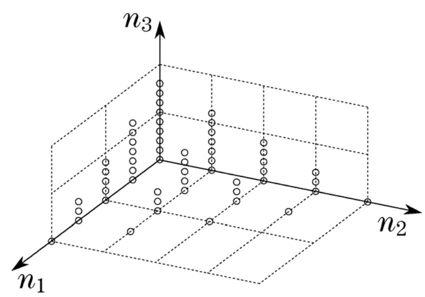

Fig. 7. Feasibility region $\Omega_{\mathrm{FR}}^{(2)}$ determined by the parameters in Table II.

of the optimal threshold has to satisfy Proposition III.3. In Scenario 2, the only parameter that changes is the revenue rate, given in this case by $R = 7$. Then, $R \leq x_1(0,8)$ and $1/R \leq x_2(0,1)$, so we cannot conclude by Theorem III.9 about the shape of Ω^{o}. Similarly, in Scenario 3 the parameters were chosen in order to satisfy the assumptions of Theorem IV.8(i) for $k = 2$ and 3 (so the resulting optimal policy is threshold type-1), whereas in Scenario 4, none of the assumptions of Theorem IV.8 is satisfied, so we cannot conclude by Theorem IV.8. For Scenarios 1 and 2, Table IV shows a lower bound on the number of all CC policies, the number of policies that satisfy: Proposition III.3, Propositions III.3 and III.2, and Propositions III.2–III.4. Table IV allows checking how the number of candidate optimal CC policies decreases as the necessary optimality conditions are added. In Scenario 1, also Theorem III.9(i) can be applied. A similar comparison is then made for Scenarios 3 and 4 in Table V using the corresponding results available for the multidimensional case.

It is important to remark that when the associated conditions of Theorem III.9 or IV.8 are not met, the optimal CC policy may not be of threshold type; in particular, it may be represented by a nonconvex set Ω^{o} [23].

B. Changes in the Optimal CC Policies Due to Traffic/Revenue Changes

Using the values of $\lambda_1, \lambda_2, \mu_1, \mu_2$ of Scenario 1, Fig. 8 shows the dependence of the shape of the optimal CC policy Ω^{o} on the revenue ratio R. Fig. 9 shows how the optimal CC policy varies when we consider different values of the arrival rates. We take $\Omega_{\mathrm{FR}}^{(1)}$ with $\mu_1 = 0.5$, $\mu_2 = 5$, and $R = 4$. The values of λ_1 and λ_2 vary in the interval [0.5, 5].

VI. FINAL DISCUSSION AND COMPARISONS TO PREVIOUS RESULTS

We have derived structural properties of the corner points of the optimal CC policies in CAC problems with nonlinearly constrained feasibility regions and K classes of users. These

TABLE IV
NUMBER OF CC POLICIES SATISFYING VARIOUS CONSTRAINTS COMING FROM THE OPTIMALITY CONDITIONS STATED IN SECTION III-A

Number of CC policies		
Propositions	**Scenario 1**	**Scenario 2**
All policies	> 98279	
Policies satisfying Proposition III.3	4861	
Policies satisfying Propositions III.3 and III.2	3432	
Policies satisfying Propositions III.3, III.2, and III.4	1023	
Policies satisfying Theorem III.9(i) (Threshold Type-1)	9	not applicable

TABLE V
NUMBER OF CC POLICIES SATISFYING VARIOUS CONSTRAINTS COMING FROM THE OPTIMALITY CONDITIONS STATED IN SECTIONS III-A AND IV

Number of CC policies		
Propositions	**Scenario 3**	**Scenario 4**
All policies	> 3958513	
Policies satisfying Proposition IV.7	156250	
Policies satisfying Propositions IV.7 and III.2	153753	
Policies satisfying Proposition IV.7 and Theorem IV.8(i) for $k = 2$ and 3 (Threshold Type-1)	5	not applicable

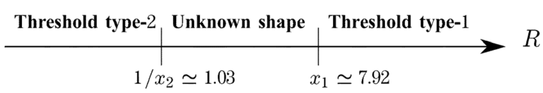

Fig. 8. Dependence of the shape of the optimal CC policy Ω° on the revenue ratio R.

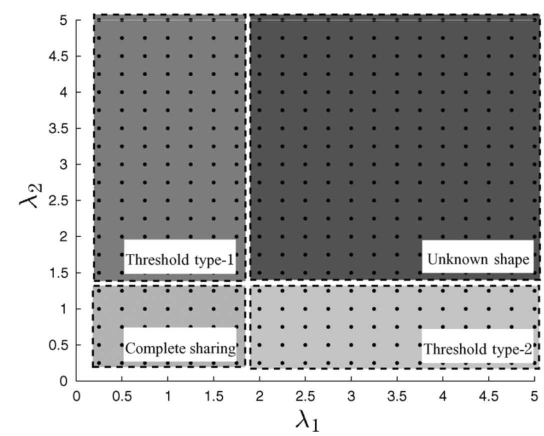

Fig. 9. Dependence of the shape of the optimal CC policy Ω° on the arrival rates λ_1 and λ_2.

properties can be exploited to narrow the search for the optimal CC policies. In particular, Theorems III.9 and IV.8 provide sufficient conditions for the optimality of several kinds of threshold-type policies and for the optimality of the complete sharing policy. Propositions III.3 and IV.7 also provide possible locations of the corner points of an optimal CC policy (in particular, when there exists an optimal threshold-type policy, they restrict the search for an optimal location of the threshold). Proposition III.2 (which holds also for the case $K > 2$) further restricts the locations of the corner points of an optimal CC policy. Although the results of the paper hold for any $K \geq 2$, to ease its readability, the case $K = 2$ was first considered, then the main results were extended to $K > 2$ classes.

The case of homogeneous Poisson arrivals and exponential call durations, which was considered in the simulations, is often assumed in the literature (see, e.g., [29]). However, the results of the paper hold also for non-Poisson arrivals (i.e., arrivals with rates depending on the state of the CAC system) and may be extended to rates also slowly depending on time. Priority among the classes of users can be incorporated in the model by choosing suitable revenues r_k [25] or by using cooperative game theory [30]. Also, trunk reservation policies can be used to this aim, but they are not CC policies. In [20, Sec. IV-A and IV-C], for a similar[4] CAC model with nonlinearly constrained feasibility regions, the authors derived a sufficient condition for which complete sharing (CS)—also called *greedy* policy in [20]—is an optimal CC policy. For $K = 2$, the sufficient condition for the optimality of CS given in [20, Equation (34)] is (rewritten using our notation)

$$\rho_1 := \frac{\lambda_1}{\mu_1} \leq R \leq \frac{\mu_2}{\lambda_2} := \frac{1}{\rho_2}. \tag{12}$$

The sufficient condition for the optimality of CS, which follows by our Theorem III.9(iii), is instead

$$\frac{1}{L_1} \leq x_1(0, B_1) < R < \frac{1}{x_2(0, B_2)} \leq L_2. \tag{13}$$

The following relationships hold.

Proposition VI.1:

$$\frac{1}{L_1} \leq x_1(0, B_1) < \rho_1 < R < \frac{1}{\rho_2} < \frac{1}{x_2(0, B_2)} \leq L_2.$$

Proposition VI.1 implies that our sufficient condition (13) for the optimality of CS is less restrictive than condition (12).

In [20, Sec. IV-C], the following sufficient condition for the optimality of CS for a feasibility region with $K > 2$ classes is given

$$r_k \geq \sum_{j \in \mathcal{K} \setminus \{k\}} r_j \frac{\lambda_j}{\mu_j} \qquad \forall k \in \mathcal{K}. \tag{14}$$

As a particularly simple but illustrative example, (14) holds, e.g., when all the ratios λ_j/μ_j are sufficiently small. On the

[4]The only significant difference between the two models has to do with the discretizations of the time variable in [20] and, consequently, of the arrival and departure processes. Our model can be considered as the limit of the one in [20] when the sampling interval tends to 0. A second minor difference is that the optimization problem in [20, Section IV-A] has a finite horizon. However, in [20, Section IV-C] the authors show that their solution solves also a discrete-time version of the infinite-horizon optimization problem with average reward per-unit time defined by the objective (1).

other hand, the sufficient condition from our Theorem IV.8(ii) is

$$r_k > \frac{\sum_{j \in \mathcal{K}\setminus\{k\}} r_j x_j\left(0, n_{j,\max}^{\mathrm{FR}}\right)}{\mathbf{C}_{G,k} - x_k\left((\mathbf{C}-\mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k}-1\right)} \tag{15}$$

for all $k \in \mathcal{K}$ and all the vectors $\mathbf{C}_G$ with $\mathbf{C}_{G,k} \neq 0$.

Proposition VI.2: For all $k \in \mathcal{K}$ and all the vectors $\mathbf{C}_G$ with $\mathbf{C}_{G,k} \neq 0$, one has

$$\frac{\sum_{j \in \mathcal{K}\setminus\{k\}} r_j x_j\left(0, n_{jd,\max}^{\mathrm{FR}}\right)}{\mathbf{C}_{G,k} - x_k\left((\mathbf{C}-\mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k}-1\right)} < \sum_{j \in \mathcal{K}\setminus\{k\}} r_j \frac{\lambda_j}{\mu_j}.$$

Proposition VI.2 implies that the sufficient conditions expressed in (14) are more restrictive than ours in (15).

When the conditions of Theorems III.9 and IV.8 do not hold, we first observe that our results still show that the search for an optimal CC policy can be restricted to CC policies with all their corner points on the grid G. Algorithm 1 provides a way to enumerate all such policies. In the case in which the number of such policies is large, it is still possible to replace the feasibility region Ω_{FR} with feasibility regions $\Omega'_{\mathrm{FR}} \subset \Omega_{\mathrm{FR}}$ and $\Omega''_{\mathrm{FR}} \supset \Omega_{\mathrm{FR}}$ with "simpler" contours (in the sense that the associated grids G' and G'' have smaller sizes than the one of G), then to evaluate the performance of all the CC policies generated by Algorithm 1 applied to Ω'_{FR} and Ω''_{FR} instead of Ω_{FR}. In this way, one obtains, respectively, a lower bound and an upper bound on the performance of an optimal CC policy for the original problem associated with Ω_{FR}. Another possible application of our results to the investigation of suboptimal policies is in the development of greedy algorithms, such as the one described in [26]. Some preliminary results in this direction were presented in [31].

A second way of restricting the search for an optimal CC policy consists in applying suitable variations of Theorems III.9 and IV.8, which show that for certain values of the parameters r_k, some corner points of the grid G can be excluded from the search of an optimal CC policy. For simplicity of exposition, in the following we only state one of such possible variations of Theorem IV.8, which provides, for $K > 2$, some steps toward the characterization of the optimality of the restricted complete sharing policy, as defined in [32] (due to space limits, we do not state an analogous result for the case $K = 2$, for which, however, one can show that the restricted complete sharing policy is a particular case of a threshold type-1 or threshold type-2 policy). Inspection of such a definition shows that, for all k classes with the exception of at most one value of k, the restricted complete sharing policy has corner points whose kth coordinates are equal to 0 or 1.[5] Interestingly, while [32] provides bounds and numerical results on the performance of the restricted complete sharing policy (only for linearly constrained feasibility regions), it does not provide conditions for its optimality.[6]

[5]The definition of the restricted complete sharing policy used here differs slightly from that used in [32]. Therein, also a way to determine for which values of k the kth coordinate is equal to 0, for which it is equal to 1, and for which it is free from these two constraints is provided. Since such conditions do not influence the form of the restricted complete sharing policy, we have not taken them into account in the following analysis in order to simplify the statement of Proposition VI.3. Of course, one might still take them into account and state a suitable variation of Proposition VI.3 at the expense of a heavier notation.

TABLE VI
PARAMETERS FOR THE APPLICATION OF PROPOSITION VI.3. IN THE TABLE, WE ASSUME THAT EACH $\lambda_k(\cdot)$ HAS A CONSTANT VALUE $\lambda_k > 0$ AND THAT THE FEASIBILITY REGION Ω_{FR} IS CHOSEN IN SUCH A WAY THAT THE POSSIBLE VALUES OF $\mathbf{C}_{G,4}$ ARE 0, 1, AND 5

r_k	ρ_k	$n_{k,\max}^{FR}$
$r_1 = 4$	$\rho_1 = 1/3$	$n_{1,\max}^{FR} = 3$
$r_2 = 3$	$\rho_2 = 1/4$	$n_{2,\max}^{FR} = 3$
$r_3 = 2$	$\rho_3 = 1/5$	$n_{3,\max}^{FR} = 3$
$r_4 = 1$	$\rho_4 = 1/6$	$n_{4,\max}^{FR} \geq 5$

Proposition VI.3: Let $K > 2$, $\lambda_1(\cdot), \lambda_2(\cdot), \ldots, \lambda_K(\cdot)$ be nonincreasing. For all but one class k, let one of the following conditions hold.

(i) For all vectors $\mathbf{C}_G$ with $\mathbf{C}_{G,k} \neq 0$ that are associated with points in the grid G, one has

$$r_k > \frac{\sum_{j \in \mathcal{K}\setminus\{k\}} r_j x_j\left(0, n_{j,\max}^{\mathrm{FR}}\right)}{\mathbf{C}_{G,k} - x_k\left((\mathbf{C}-\mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k}-1\right)}. \tag{16}$$

(ii) There exists a point $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_K) \in G$ such that $\alpha_k = 1$

$$r_k \leq \sum_{j \in \mathcal{K}\setminus\{k\}} r_j x_j\left(0, n_{j,\max}^{\mathrm{FR}}\right) \tag{17}$$

and for all vectors $\mathbf{C}_G$ with $\mathbf{C}_{G,k} \neq 0, 1$ that are associated with points in the grid G, one has

$$r_k > \frac{\sum_{j \in \mathcal{K}\setminus\{k\}} r_j x_j\left(0, n_{j,\max}^{\mathrm{FR}}\right)}{\mathbf{C}_{G,k} - x_k\left((\mathbf{C}-\mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k}-1\right)}. \tag{18}$$

Then, any optimal CC policy Ω° has the form of the restricted complete sharing policy.

As an example, Proposition VI.3 can be applied to the set of parameters reported in Table VI for $K = 4$. In this particular example, due to Proposition VI.3, the number of possible optimal corner points is bounded from above by only $2(n_{3,\max}^{\mathrm{FR}}+1) = 8$.

VII. PROOFS AND TECHNICAL LEMMAS

Proposition III.2: The following definition and proposition are needed in the proof. Note that Definition VII.1 holds also for the case of $K > 2$ classes of users.

Definition VII.1: A nonempty set $S^- \subsetneq \Omega_{\mathrm{FR}}$ is incrementally removable with respect to a CC set $\Omega \subseteq \Omega_{\mathrm{FR}}$ (IR_Ω) if and only if $S^- \subsetneq \Omega$ and $\Omega \setminus S^-$ is still a CC set. A nonempty set $S^+ \subsetneq \Omega_{\mathrm{FR}}$ is incrementally admissible with respect to Ω (IA_Ω) if and only if $S^+ \cap \Omega = \emptyset$ and $\Omega \cup S^+$ is still a CC set.

Proposition VII.2 [26, Proposition III.3]: Let $K = 2$, (α, β) be a type-2 corner point for Ω and suppose that there exist $n, m, p \in \mathbb{N}_0$ such that $S^- := \{(\alpha - 1 - j, \beta + i) : j = 0, \ldots, n, i = 0, \ldots, p\} \subsetneq \Omega$, is IR_Ω, and $S^+ := \{(\alpha + s, \beta + i) : s = 0, \ldots, m, i = 0, \ldots, p\} \subsetneq \Omega_{\mathrm{FR}}$, is IA_Ω. Then, at least

[6]There are cases in which the restricted complete sharing policy is strictly suboptimal. Indeed, for $K > 2$, simulation results showing significant improvements of certain multiple-threshold policies over the restricted complete sharing policy are presented in [14, Table IV] (the table refers to the case $K = 16$).

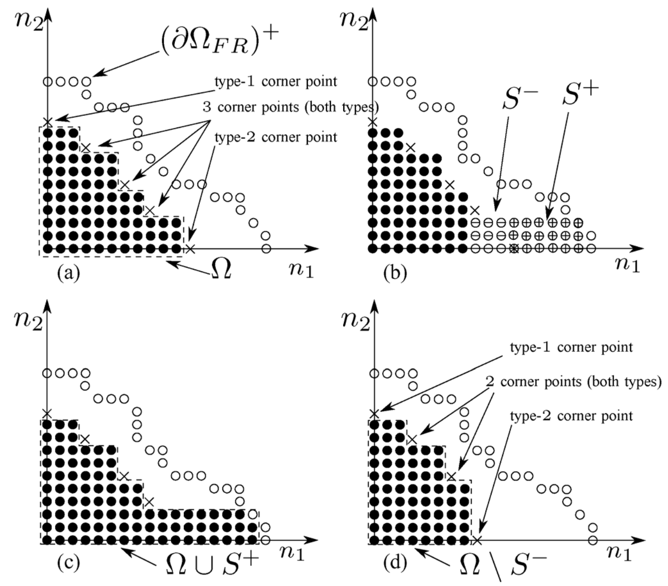

Fig. 10 Sets Ω, S^+, and S^- used in the proof of Proposition III.2.

one of the following inequalities holds: 1) $J(\Omega \cup S^+) > J(\Omega)$; 2) $J(\Omega \setminus S^-) > J(\Omega)$.

Proof of Proposition III.2 for $K = 2$: Let us consider any CC set $\Omega \subseteq \Omega_{\mathrm{FR}}$ such that $\Omega \cap (\partial\Omega_{\mathrm{FR}})^+ = \emptyset$. We show that by a repeated application of Proposition VII.2, one can find a sequence of CC sets associated with better CC policies, such that at least one of these sets intersects $(\partial\Omega_{\mathrm{FR}})^+$.

Let $I(\Omega)$ be the set whose elements are the corner points of Ω and $|I(\Omega)|$ its cardinality. We observe that for any two such successive corner points $(\alpha^{(i)}, \beta^{(i)})$ and $(\alpha^{(i+1)}, \beta^{(i+1)})$, the coordinate-convexity of Ω implies $\beta^{(i)} > \beta^{(i+1)}$. As $\Omega \cap (\partial\Omega_{\mathrm{FR}})^+ = \emptyset$, Ω has at least two corner points, where the first one $(\alpha^{(1)} = 0, \beta^{(1)} > 0)$ is on the n_2-axis and the last one $(\alpha^{(|I(\Omega)|)} > 0, \beta^{(|I(\Omega)|)} = 0)$ is on the n_1-axis [see Fig. 10(a)].

a) If $|I(\Omega)| > 2$, then we apply Proposition VII.2 to the corner point $(\alpha^{(|I(\Omega)|)} > 0, \beta^{(|I(\Omega)|)} = 0)$, choosing $n = \alpha^{(|I(\Omega)|)} - \alpha^{(|I(\Omega)|-1)} - 1$, $p = \beta^{(|I(\Omega)|-1)} - \beta^{(|I(\Omega)|)} - 1$, and m the largest nonnegative integer such that $(\alpha^{(|I(\Omega)|)} + m, \beta^{(|I(\Omega)|-1)} - \beta^{(|I(\Omega)|)} - 1) \in (\partial\Omega_{\mathrm{FR}})^+$ [see Fig. 10(b)]. By Proposition VII.2, at least one of the inequalities $J(\Omega \setminus S^-) > J(\Omega)$ and $J(\Omega \cup S^+) > J(\Omega)$ holds. By construction, $(\Omega \cup S^+) \cap (\partial\Omega_{\mathrm{FR}})^+ \neq \emptyset$, so if $J(\Omega \cup S^+) > J(\Omega)$, the statement is proved [see Fig. 10(c)]. Otherwise, $J(\Omega \setminus S^-) > J(\Omega)$. Note that $\Omega \setminus S^-$ does not intersect $(\partial\Omega_{\mathrm{FR}})^+$ and has only $|I(\Omega)| - 1$ corner points [see Fig. 10(d)], where the last one is $(\alpha^{(|I(\Omega)|-1)}, 0)$, where $\alpha^{(|I(\Omega)|-1)} > 0$. Thus, we can repeat the arguments used above starting from $\Omega \setminus S^-$ instead of Ω. After at most $|I(\Omega)| - 2$ applications of Proposition VII.2, we reach one of the following cases: Either we find a CC policy better than the initial one and with associated CC set intersecting $(\partial\Omega_{\mathrm{FR}})^+$, or we end up with the next case (b).

b) If $|I(\Omega)| = 2$, then Ω is rectangular. Hence, the CAC system performs a decoupling between the two classes of users. Then, Ω can be improved by extending one of its opposite sides until it meets $(\partial\Omega_{\mathrm{FR}})^+$. ■

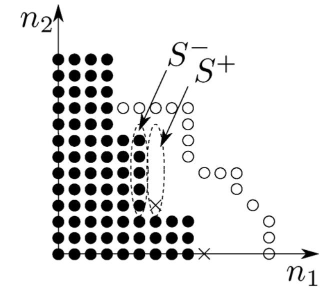

Fig. 11. Example of a CC set Ω having a type-2 corner point (α, β) for which $\alpha \neq l_1^{\Omega_{\mathrm{FR}}}(j_2) + 1$ for every $j_2 = 1, \ldots, n_{2,\max}^{\mathrm{FR}}$.

Proposition III.3: We recall from [2] that the definition of the objective $J(\cdot)$ in (1) can be extended consistently to all (not necessarily CC) sets $S \subseteq \Omega_{\mathrm{FR}}$ in the following way:

$$J(S) := \frac{H(S)}{G(S)} \tag{19}$$

with

$$H(S) := \sum_{\mathbf{n} \in S} (\mathbf{n} \cdot \mathbf{r}) \prod_{k=1}^{2} q_k(n_k) \tag{20}$$

$$G(S) := \sum_{\mathbf{n} \in S} \prod_{k=1}^{2} q_k(n_k). \tag{21}$$

For a rectangular region $S := \{a, a+1, \ldots, b\} \times \{c, c+1, \ldots, d\}$, from (4), (20), and (21), it follows that

$$J(S) = r_1 x_1(a, b) + r_2 x_2(c, d). \tag{22}$$

Lemma VII.3 [2, Lemma 2], Extended to the K-Dimensional Case: Let Ω^{o} be an optimal CC policy. Then: i) if S is $IA_{\Omega^{\mathrm{o}}}$, then $J(S) \leq J(\Omega^{\mathrm{o}})$; ii) if S is $IR_{\Omega^{\mathrm{o}}}$, then $J(S) \geq J(\Omega^{\mathrm{o}})$.

Proof of Lemma VII.3: i) Let S be IA_{Ω}^{o}. By the definition of $J(S)$ and the optimality of Ω^{o}, we have

$$\frac{H(\Omega^{\mathrm{o}}) + H(S)}{G(\Omega^{\mathrm{o}}) + G(S)} = J(\Omega^{\mathrm{o}} \cup S) \leq J(\Omega^{\mathrm{o}}) = \frac{H(\Omega^{\mathrm{o}})}{G(\Omega^{\mathrm{o}})}$$

which in turn implies

$$J(S) = \frac{H(S)}{G(S)} \leq \frac{H(\Omega^{\mathrm{o}})}{G(\Omega^{\mathrm{o}})} = J(\Omega^{\mathrm{o}}).$$

The proof of ii) is similar.

Proof of Proposition III.3: We prove only (i), as (ii) is obtained in the same way by exchanging the roles of the two classes of users. Suppose that (7) is violated for every $j = 1, \ldots, n_{2,\max}^{\mathrm{FR}}$. By choosing $n = l_2^{\Omega^{\mathrm{o}}}(\alpha - 1) - \beta \geq 0$, $S^-(n) = \{(\alpha-1, \beta+i) : i = 0, \ldots, n\} \subseteq \Omega^{\mathrm{o}}$, and $S^+(n) = \{(\alpha, \beta+i) : i = 0, \ldots, n\} \subseteq \Omega_{\mathrm{FR}} \setminus \Omega^{\mathrm{o}}$ (see Fig. 11), it follows that the sets $\Omega^{\mathrm{o}} \setminus S^-(n)$ and $\Omega^{\mathrm{o}} \cup S^+(n)$ are CC, so $S^-(n)$ is $IR_{\Omega^{\mathrm{o}}}$ and $S^+(n)$ is $IA_{\Omega^{\mathrm{o}}}$. By (22), we get $J(S^-(n)) = r_1(\alpha - 1) + r_2 x_2(\beta, \beta + n) < r_1 \alpha + r_2 x_2(\beta, \beta + n) = J(S^+(n))$, but this contradicts the optimality condition in Lemma VII.3, so there exists $j_2 = 1, \ldots, n_{2,\max}^{\mathrm{FR}}$ such that (7) holds. ■

Proof of Proposition III.5 for $K = 2$: We build Ω starting from Ω_{FR} and removing subregions from Ω_{FR}, each associated

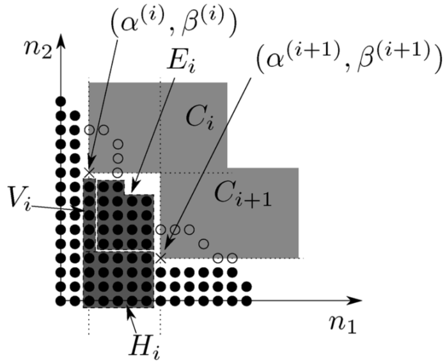

Fig. 12. Sets V_i, H_i, and E_i considered in the proof of Proposition III.5.

with one of the $|I(\Omega)|$ corner points of Ω. To simplify the notations, we assume $I(\Omega) \neq \emptyset$. The case $I(\Omega) = \emptyset$ can be dealt with similarly.

Let $(\alpha^{(i)}, \beta^{(i)})$ be one of these corner points. Consider a point $(\hat{n}_1, \hat{n}_2) \in C_i^- := \{\mathbf{n} \in \Omega_{\mathrm{FR}} : n_1 \geq \alpha^{(i)} \text{ and } n_2 \geq \beta^{(i)}\}$ and suppose that $(\hat{n}_1, \hat{n}_2) \in \Omega$. The coordinate-convexity of Ω implies $(\hat{n}_1, \beta^{(i)}) \in \Omega$ and $(\alpha^{(i)}, \beta^{(i)}) \in \Omega$, which is a contradiction. Therefore, each corner point $(\alpha^{(i)}, \beta^{(i)})$ excludes from Ω_{FR} the set C_i^-. Then

$$\Omega \subseteq \left(\Omega_{\mathrm{FR}} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-\right). \tag{23}$$

Now, we show that also $\Omega = (\Omega_{\mathrm{FR}} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-)$ holds, so Ω is completely determined by the knowledge of the locations of all its corner points. This can be proved by showing that if $(\Omega_{\mathrm{FR}} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \setminus \Omega \neq \emptyset$, then Ω would have at least $|I(\Omega)|+1$ corner points, which is a contradiction. We detail this part of the proof considering what happens inside the strip $Q_i := \{(n_1, n_2) \in \mathbb{N}_0^2 : \alpha^{(i)} \leq n_1 < \alpha^{(i+1)}\}$ between two consecutive corner points $(\alpha^{(i)}, \beta^{(i)})$ and $(\alpha^{(i+1)}, \beta^{(i+1)})$ (suppose for simplicity of notation that all their coordinates are positive).[7] First of all, note that $(\Omega_{\mathrm{FR}} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \cap Q_i = (\Omega_{\mathrm{FR}} \setminus C_i^-) \cap Q_i$. Then, by the definition of a corner point and the coordinate-convexity of Ω one has $V_i := \{(\alpha^{(i)}, k) : k = 0, \ldots, \beta^{(i)} - 1\} \subseteq \Omega$ and $H_i := \{(\alpha^{(i)} + h, j) : h = 0, \ldots, \alpha^{(i+1)} - \alpha^{(i)} - 1, j = 0, \ldots, \beta^{(i+1)}\} \subseteq \Omega$.

Let $E_i := ((\Omega_{\mathrm{FR}} \setminus C_i^-) \cap Q_i) \setminus (V_i \cup H_i)$, $F_i := E_i \setminus \Omega$, and suppose that $F_i \neq \emptyset$. Let (p_1, p_2) be one of the points of F_i with minimal first coordinate, then by the coordinate-convexity of Ω one has $\{(p_1 - 1, l) : l = 0, \ldots, p_2\} \subseteq \Omega$. Let $(p_1, \hat{p}_2)$ be the only point of F_i with first coordinate p_1 and minimal second coordinate. One can check by the definition that $(p_1, \hat{p}_2)$ is a corner point, but this is a contradiction since the only corner point in the strip Q_i is $(\alpha^{(i)}, \beta^{(i)})$ by construction. Then, F_i must be empty, and therefore $E_i \subseteq \Omega$. Fig. 12 shows an example of the sets V_i, H_i, and E_i considered in this proof

Summing up, $(\Omega_{\mathrm{FR}} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \cap Q_i \subseteq \Omega \cap Q_i$. This, combined with $\Omega \cap Q_i \subseteq (\Omega_{\mathrm{FR}} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \cap Q_i$ [obtained from (23)], proves that $\Omega \cap Q_i = (\Omega_{\mathrm{FR}} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \cap Q_i$. Similarly, for the sets Q_0 and $Q_{|I(\Omega)|}$ defined in footnote 7, one has $\Omega \cap Q_0 = (\Omega_{\mathrm{FR}} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \cap Q_0$, and $\Omega \cap Q_{|I(\Omega)|+1} = (\Omega_{\mathrm{FR}} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-) \cap Q_{|I(\Omega)|+1}$. Hence, $\Omega = (\Omega_{\mathrm{FR}} \setminus \cup_{i=1}^{|I(\Omega)|} C_i^-)$. ■

[7]Similarly, if $\alpha^{(1)} > 0$, then one should also consider the first strip $Q_0 := \{(n_1, n_2) \in \mathbb{N}_0^2 : 0 \leq n_1 < \alpha^{(1)}\}$, and if $\beta^{(|I(\Omega)|)} > 0$, then one should take into account also the last strip $Q_{|I(\Omega)|+1} := \{(n_1, n_2) \in \mathbb{N}_0^2 : n_1 \geq \alpha^{(|I(\Omega)|)}\}$.

Proof of Theorem III.9: The proof of Theorem III.9 is obtained combining the following technical lemmas. Lemma VII.4 can be also proved as a consequence of Proposition III.5.

Lemma VII.4 (From [2, Lemma 1]): Let $K = 2$. A CC policy is a type-k threshold if and only if it has no type-k corner points $(k = 1, 2)$.

Lemma VII.5 (From [2, Lemma 3]): For any nonnegative integers a, b, c, d, e, f with $b \geq a, d \geq c, f \geq e, b+d \geq f, a+c \geq e$, we have $x_k(a, b) + x_k(c, d) \geq x_k(e, f)$; $k = 1, \ldots, K$.

Lemma VII.6 is our extension of [2, Lemma 4] to general nonlinearly constrained feasibility regions. With respect to [2], due to the different shape of the feasibility region, in general, it is not true that $j \neq h$ implies $l_1^{\Omega_{\mathrm{FR}}}(j) \neq l_1^{\Omega_{\mathrm{FR}}}(h)$. As shown in Fig. 2, for every $j_2 \in \{0, \ldots, n_{2,\max}^{\mathrm{FR}}\}$, there exist a minimum index $j_2^{(l)} \leq j_2$ and a maximum index $j_2^{(u)} \geq j_2$ such that $l_1^{\Omega_{\mathrm{FR}}}(\cdot)$ is constant on the set $\{j_2^{(l)}, \ldots, j_2^{(u)}\} \subseteq \{0, \ldots, n_{2,\max}^{\mathrm{FR}}\}$. Similarly, for every $j_1 \in \{0, \ldots, n_{1,\max}^{\mathrm{FR}}\}$, there exist a minimum index $j_1^{(l)} \leq j_1$ and a maximum index $j_1^{(u)} \geq j_1$ such that $l_2^{\Omega_{\mathrm{FR}}}(\cdot)$ is constant on the set $\{j_1^{(l)}, \ldots, j_1^{(u)}\} \subseteq \{0, \ldots, n_{1,\max}^{\mathrm{FR}}\}$. Let $B_1 := \max\{j_1^{(u)} - j_1^{(l)} + 1 : j_1 = 0, \ldots, n_{1,\max}^{\mathrm{FR}}\} \leq n_{1,\max}^{\mathrm{FR}} + 1$ and $B_2 := \max\{j_2^{(u)} - j_2^{(l)} + 1 : j_2 = 0, \ldots, n_{2,\max}^{\mathrm{FR}}\} \leq n_{2,\max}^{\mathrm{FR}} + 1$.

Lemma VII.6:

(i) If (α, β) is a type-2 corner point for Ω^{o} and $\lambda_2(\cdot)$ is nonincreasing, then for some $j_2 = 1, \ldots, n_{2,\max}^{\mathrm{FR}}$, (7) holds together with

$$\begin{aligned} R x_2(0, B_2) \geq\, & x_1\left(l_1^{\Omega_{\mathrm{FR}}}\left(j_2^{(l)}\right) + 1, l_1^{\Omega_{\mathrm{FR}}}\left(j_2^{(l)} - 1\right)\right) \\ & - x_1\left(l_1^{\Omega_{\mathrm{FR}}}\left(j_2^{(u)} + 1\right) + 1, l_1^{\Omega_{\mathrm{FR}}}\left(j_2^{(u)}\right)\right). \end{aligned} \tag{24}$$

(ii) If (α, β) is a type-1 corner point for Ω^{o} and $\lambda_1(\cdot)$ is nonincreasing, then for some $j_1 = 1, \ldots, n_{1,\max}^{\mathrm{FR}}$, (8) holds together with

$$\begin{aligned} \frac{1}{R} x_1(0, B_1) \geq\, & x_2\left(l_2^{\Omega_{\mathrm{FR}}}\left(j_1^{(l)}\right) + 1, l_2^{\Omega_{\mathrm{FR}}}\left(j_1^{(l)} - 1\right)\right) \\ & - x_2\left(l_2^{\Omega_{\mathrm{FR}}}\left(j_1^{(u)} + 1\right) + 1, l_2^{\Omega_{\mathrm{FR}}}\left(j_1^{(u)}\right)\right). \end{aligned} \tag{25}$$

Proof of Lemma VII.6: Given a type-2 corner point (α, β), by Proposition III.3(i) for some $j_2 = 1, \ldots, n_{2,\max}^{\mathrm{FR}}$, one has $\alpha = l_1^{\Omega_{\mathrm{FR}}}(j_2) + 1$. Choosing $n = l_2^{\Omega^{\mathrm{o}}}(\alpha - 1) - \beta \geq 0$, $m = \max\{(l_2^{\Omega_{\mathrm{FR}}}(\alpha) - \beta), 0\}$, $\hat{S}^-(n) = \{l_1^{\Omega_{\mathrm{FR}}}(j_2^{(u)} + 1) + 1, \ldots, l_1^{\Omega_{\mathrm{FR}}}(j_2^{(u)})\} \times \{\beta, \ldots, \beta + n\} \subseteq \Omega^{\mathrm{o}}$, and $\hat{S}^+(m) = \{l_1^{\Omega_{\mathrm{FR}}}(j_2^{(l)}) + 1, \ldots, l_1^{\Omega_{\mathrm{FR}}}(j_2^{(l)} - 1)\} \times \{\beta, \ldots, \beta + m\} \subseteq \Omega_{\mathrm{FR}} \setminus \Omega^{\mathrm{o}}$ (see Fig. 13), it follows that the sets $\Omega^{\mathrm{o}} \setminus \hat{S}^-(n)$ and $\Omega^{\mathrm{o}} \cup \hat{S}^+(m)$ are CC, so $\hat{S}^-(n)$ is $IR_{\Omega^{\mathrm{o}}}$ and $\hat{S}^+(m)$ is $IA_{\Omega^{\mathrm{o}}}$. By (22) one gets

$$\begin{aligned} J\left(\hat{S}^-(n)\right) =\, & r_1 x_1\left(l_1^{\Omega_{\mathrm{FR}}}\left(j_2^{(u)} + 1\right) + 1, l_1^{\Omega_{\mathrm{FR}}}\left(j_2^{(u)}\right)\right) \\ & + r_2 x_2(\beta, \beta + n) \end{aligned}$$

and

$$\begin{aligned} J\left(\hat{S}^+(m)\right) =\, & r_1 x_1\left(l_1^{\Omega_{\mathrm{FR}}}\left(j_2^{(l)}\right) + 1, l_1^{\Omega_{\mathrm{FR}}}\left(j_2^{(l)} - 1\right)\right) \\ & + r_2 x_2(\beta, \beta + m). \end{aligned}$$

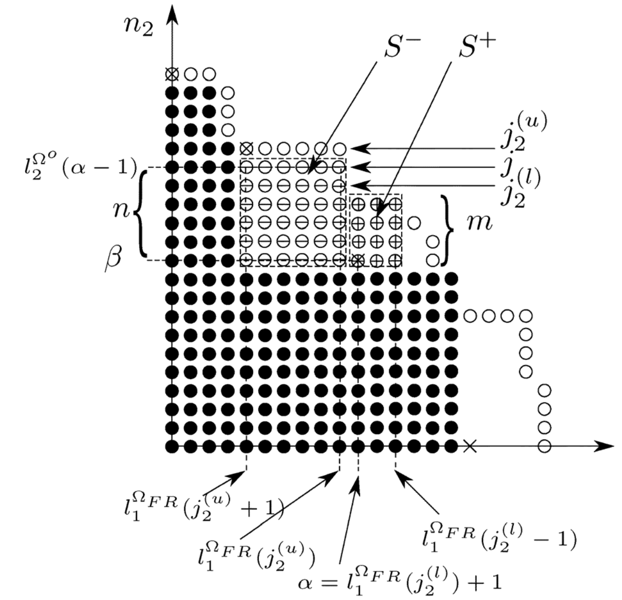

Fig. 13. Description of a step in the proof of Lemma VII.6.

Combining these equalities with Lemma VII.3 (which implies $J(\hat{S}^-(n)) \geq J(\hat{S}^+(n))$), one has

$$\begin{aligned} &R\left(x_2(\beta,\beta+n) - x_2(\beta,\beta+m)\right) \\ &\quad\geq x_1\left(l_1^{\Omega_{\rm FR}}\left(j_2^{(l)}\right)+1, l_1^{\Omega_{\rm FR}}\left(j_2^{(l)}-1\right)\right) \\ &\qquad - x_1\left(l_1^{\Omega_{\rm FR}}\left(j_2^{(u)}+1\right)+1, l_1^{\Omega_{\rm FR}}\left(j_2^{(u)}\right)\right). \end{aligned} \tag{26}$$

Since $0 \leq n-m \leq B_2$ (see Fig. 13) and $\lambda_2(\cdot)$ is nonincreasing, by Lemma VII.5, one obtains

$$x_2(0,B_2) \geq x_2(\beta,\beta+n) - x_2(\beta,\beta+m)$$

which, when combined with (26), proves (24). Formula (ii) is proved similarly by exchanging the two classes os users. ■

Lemma VII.7: Let $\lambda_k(\cdot)$ be nonincreasing for $k=1,2$.

(i) If $1/R < L_1$, where

$$L_1 := \min_{j_1=1,\dots,n_{1,\max}^{\rm FR}} \left\{ \frac{x_2\left(l_2^{\Omega_{\rm FR}}\left(j_1^{(l)}\right)+1, l_2^{\Omega_{\rm FR}}\left(j_1^{(l)}-1\right)\right)}{x_1(0,B_1)} - \frac{x_2\left(l_2^{\Omega_{\rm FR}}\left(j_1^{(u)}+1\right)+1, l_2^{\Omega_{\rm FR}}\left(j_1^{(u)}\right)\right)}{x_1(0,B_1)} \right\}$$

then $\Omega^{\rm o}$ is threshold type-1, and the threshold is equal to some $l_1^{\Omega_{\rm FR}}(j_2)$ for some $j_2 = 0,\dots,n_{2,\max}^{\rm FR}$.

(ii) If $R < L_2$, where

$$L_2 := \min_{j_2=1,\dots,n_{2,\max}^{\rm FR}} \left\{ \frac{x_1\left(l_1^{\Omega_{\rm FR}}\left(j_2^{(l)}\right)+1, l_1^{\Omega_{\rm FR}}\left(j_2^{(l)}-1\right)\right)}{x_2(0,B_2)} - \frac{x_1\left(l_1^{\Omega_{\rm FR}}\left(j_2^{(u)}+1\right)+1, l_1^{\Omega_{\rm FR}}\left(j_2^{(u)}\right)\right)}{x_2(0,B_2)} \right\}$$

then $\Omega^{\rm o}$ is threshold type-2, and the threshold is equal to some $l_2^{\Omega_{\rm FR}}(j_1)$ for some $j_1 = 0,\dots,n_{1,\max}^{\rm FR}$.

(iii) If $1/L_1 < R < L_2$, then $\Omega^{\rm o} = \Omega_{\rm FR}$.

Proof of Lemma VII.7:

(i) If $1/R < L_1$, then by Lemma VII.6(ii), $\Omega^{\rm o}$ has no type-1 corner points, so it is a threshold type-1 policy by Lemma VII.4. Let t_1 denote the corresponding threshold. Then, either $t_1 = n_{1,\max}^{\rm FR} := l_1^{\Omega_{\rm FR}}(0)$ or $(t_1+1,0)$ is a type-2 corner point for $\Omega^{\rm o}$. In the second case, by Proposition III.3(i), we have $t_1 + 1 = l_1^{\Omega_{\rm FR}}(j_2)+1$ for some $j_2 = 1,\dots,n_{2,\max}^{\rm FR}$.

(ii) (ii) is proved similarly.

(iii) If $1/L_1 < R < L_2$, then by parts (i) and (ii) $\Omega^{\rm o}$ is both threshold type-1 and threshold type-2, so it coincides with $\Omega_{\rm FR}$.

■

Remark VII.8: In the case of a linearly constrained feasibility region with $B_2 = 1$ (i.e., the one considered in [2]), one has $j_1^{(l)} = j_1^{(u)}$ for each $j_1 = 0,\dots,n_{1,\max}^{\rm FR}$, and $L_1 = 1/x_1(0,B_1)$. Hence, in this case Lemma VII.7(i) reduces to [2, Theorem 1(i)].

Proof of Theorem III.9: For each $j_1 = 0,\dots,n_{1,\max}^{\rm FR}$, it follows from the definitions of $x_2(\cdot,\cdot)$ and of $j_1^{(l)}$, $j_1^{(u)}$ that

$$x_2\left(l_2^{\Omega_{\rm FR}}\left(j_1^{(l)}\right)+1, l_2^{\Omega_{\rm FR}}\left(j_1^{(l)}-1\right)\right) \geq l_2^{\Omega_{\rm FR}}\left(j_1^{(l)}\right)+1$$
$$x_2\left(l_2^{\Omega_{\rm FR}}\left(j_1^{(u)}+1\right)+1, l_2^{\Omega_{\rm FR}}\left(j_1^{(u)}\right)\right) \leq l_2^{\Omega_{\rm FR}}\left(j_1^{(u)}\right)$$

and $l_2^{\Omega_{\rm FR}}(j_1^{(u)}) = l_2^{\Omega_{\rm FR}}(j_1^{(l)})$, so $L_1 \geq 1/x_1(0,B_1)$. Similarly, we have $L_2 \geq 1/x_2(0,B_2)$. ■

Proposition IV.7: Before proving Proposition IV.7, as we did before for the two-dimensional case, we extend the definition of the objective $J(\cdot)$ to any K-dimensional region $S \subseteq \Omega_{\rm FR}$ as follows:

$$J(S) := \frac{H(S)}{G(S)} = \frac{\sum_{\mathbf{n}\in S}(\mathbf{n}\cdot\mathbf{r})\prod_{k=1}^K q_k(n_k)}{\sum_{\mathbf{n}\in S}\prod_{k=1}^K q_k(n_k)}. \tag{27}$$

Note that, for any CC set $\Omega \subseteq \Omega_{\rm FR}$ and $S_1 \subseteq \Omega \subseteq S_2$ with S_1 and S_2 hyper-rectangles, it follows from (27) that $J(S_2) \geq J(\Omega) \geq J(S_1)$ (instead, this is not true in general when S_1 and S_2 are not hyper-rectangles). Moreover, given a K-dimensional region S of the form

$$S := \left\{\mathbf{n}\in\Omega_{\rm FR} : \mathbf{n} = \mathbf{n}^{(1)} + \mathbf{n}^{(2)}, \mathbf{n}^{(1)} \in S' \text{ and } \mathbf{n}^{(2)} \in \{a\mathbf{e}_k,\dots,b\mathbf{e}_k\}\right\} \tag{28}$$

where a, $b \in \mathbb{N}_0$, $a \leq b$ and $S' \subsetneq N_0^K$ is made up of points whose kth component is 0, it follows from (27) that

$$J(S) = J(S') + r_k x_k(a,b). \tag{29}$$

Proof of Proposition IV.7: We show that if a corner point $\boldsymbol{\alpha} \notin G$ exists for an optimal CC policy $\Omega^{\rm o}$, then we can construct two CC regions $S^- \subsetneq \Omega_{\rm FR}, S^+ \subsetneq \Omega_{\rm FR}$ such that a necessary condition for the optimality of $\Omega^{\rm o}$ is violated.

By construction of the grid G (Definition IV.4), since $\boldsymbol{\alpha} \notin G$, there exists an index $k \in \mathcal{K}$ such that the component α_k of $\boldsymbol{\alpha}$ is not in the set of possible values assumed by the kth coordinate of a point of the grid. Let $\hat{\alpha}_k$ be the largest value smaller than α_k that can be assumed by the kth coordinate of a point of the grid, and $\hat{\mathbf{g}}$ a point of the grid whose kth component is $\hat{\alpha}_k$ and whose associated vector is $\hat{\mathbf{C}}_G$. Without loss of generality, we suppose that there is no other corner point $\tilde{\boldsymbol{\alpha}}$ of Ω^{o} whose kth component $\tilde{\alpha}_k$ satisfies $\hat{\alpha}_k < \tilde{\alpha}_k < \alpha_k$. Let

$$\begin{aligned} S'_{\hat{\alpha}_k} &:= \Omega \cap \mathcal{P}_k(\hat{\alpha}_k) \\ S'_{\alpha_k} &:= \Omega \cap \mathcal{P}_k(\alpha_k) \end{aligned}$$

and

$$S' := \pi_{\mathcal{K}\setminus\{k\}}\left(S'_{\hat{\alpha}_k}\right) \setminus \pi_{\mathcal{K}\setminus\{k\}}\left(S'_{\alpha_k}\right).$$

Then, the set

$$S^- := \Big\{\mathbf{n} \in \Omega_{\mathrm{FR}} : \mathbf{n} = \mathbf{n}^{(1)} + \mathbf{n}^{(2)}, \mathbf{n}^{(1)} \in S' \text{ and } \mathbf{n}^{(2)} = (\alpha_k - 1)\mathbf{e}_k\Big\} \quad (30)$$

is $IR_{\Omega^{\mathrm{o}}}$, whereas, by the construction of the grid, it follows that between $\hat{\mathbf{C}}_G$ and its consecutive point $(\hat{\mathbf{C}} + \mathbf{e}_k)_G$, the cross section of Ω_{FR} along the kth axis does not change, so

$$S^+ := \Big\{\mathbf{n} \in \Omega_{\mathrm{FR}} : \mathbf{n} = \mathbf{n}^{(1)} + \mathbf{n}^{(2)}, \mathbf{n}^{(1)} \in S' \text{ and } \mathbf{n}^{(2)} = \alpha_k\mathbf{e}_k\Big\} \quad (31)$$

is a subset of Ω_{FR} and is $IA_{\Omega^{\mathrm{o}}}$. By formula (29), we get

$$\begin{aligned} J(S^-) &= J(S') + r_k(\alpha_k - 1) \\ J(S^+) &= J(S') + r_k\alpha_k. \end{aligned}$$

Thus, one gets $J(S^-) < J(S^+)$, which contradicts the optimality condition in Lemma VII.3, and one concludes that if a corner point of Ω^{o} exists, then it has to be in the grid G. ■

Theorem IV.8: The proof of Theorem IV.8 is based on the following lemma, which is an extension of Lemma VII.7 to the multidimensional case.

Lemma VII.9: If the K-tuple $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_K) \in G$ is a corner point for an optimal CC policy Ω^{o} and is associated with the vector $\mathbf{C}_G$, and $\lambda_1(\cdot), \lambda_2(\cdot), \ldots, \lambda_K(\cdot)$ are nonincreasing, then, for all k such that $\alpha_k \neq 0$, one has

$$\begin{aligned} &\sum_{j\in\mathcal{K}\setminus\{k\}} r_j x_j\left(0, n_{j,\max}^{\mathrm{FR}}\right) \\ &\quad\geq \sum_{j\in\mathcal{K}\setminus\{k\}} r_j x_j\left(0, n_{j,\max}^{\mathrm{FR}} - \alpha_j\right) \\ &\quad\geq r_k\left(\alpha_k - x_k\left((\mathbf{C}-\mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1\right)\right). \end{aligned} \quad (32)$$

Proof of Lemma VII.9: For each of the components $\alpha_k \neq 0$ of $\boldsymbol{\alpha}$, we can construct two CC regions S_k^- and S_k^+ as follows. Likewise in the proof of Proposition IV.7, let $\hat{\alpha}_k$ be the largest value smaller than α_k that can be assumed by the kth coordinate of a point of the grid, and $\hat{\mathbf{g}}$ a point of the grid whose kth component is $\hat{\alpha}_k$ and whose associated vector is $\hat{\mathbf{C}}_G$. Let

$$\begin{aligned} S'_{\hat{\alpha}_k} &:= \Omega \cap \mathcal{P}_k(\hat{\alpha}_k) \\ S'_{\alpha_k} &:= \Omega \cap \mathcal{P}_k(\alpha_k) \\ \Omega'_{\mathrm{FR},\alpha_k} &:= \Omega_{\mathrm{FR}} \cap \mathcal{P}_k(\alpha_k) \\ S_k'^- &:= \pi_{\mathcal{K}\setminus\{k\}}\left(S'_{\hat{\alpha}_k}\right) \setminus \pi_{\mathcal{K}\setminus\{k\}}\left(S'_{\alpha_k}\right) \end{aligned}$$

and

$$S_k'^+ := S_k'^- \cap \pi_{\mathcal{K}\setminus\{k\}}\left(\Omega'_{\mathrm{FR},\alpha_k}\right).$$

Then, the set

$$S_k^- := \Big\{\mathbf{n} \in \Omega_{\mathrm{FR}} : \mathbf{n} = \mathbf{n}^{(1)} + \mathbf{n}^{(2)}, \mathbf{n}^{(1)} \in S_k'^- \text{ and } \mathbf{n}^{(2)} \in \{\hat{\alpha}_k\mathbf{e}_k, \ldots, (\alpha_k - 1)\mathbf{e}_k\}\Big\} \quad (33)$$

is clearly $IR_{\Omega^{\mathrm{o}}}$, whereas

$$S_k^+ := \Big\{\mathbf{n} \in \Omega_{\mathrm{FR}} : \mathbf{n} = \mathbf{n}^{(1)} + \mathbf{n}^{(2)}, \mathbf{n}^{(1)} \in S_k'^+ \text{ and } \mathbf{n}^{(2)} = \alpha_k\mathbf{e}_k\Big\} \quad (34)$$

is a subset of Ω_{FR} and is also $IA_{\Omega^{\mathrm{o}}}$. Using (29), we obtain

$$J\left(S_k^-\right) = J\left(S_k'^-\right) + r_k x_k\left((\mathbf{C}-\mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1\right)$$

and

$$\begin{aligned} J\left(S_k^+\right) &= J\left(S_k'^+\right) + r_k x_k(\mathbf{C}_{G,k}, \mathbf{C}_{G,k}) \\ &= J\left(S_k'^+\right) + \alpha_k r_k. \end{aligned}$$

Combining these equalities with Lemma VII.3, we have

$$\begin{aligned} J\left(S_k^-\right) &= J\left(S_k'^-\right) + r_k x_k\left((\mathbf{C}-\mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1\right) \\ &\geq J\left(S_k'^+\right) + \alpha_k r_k = J\left(S_k^+\right). \end{aligned}$$

Now, $J(S_k'^+)$ is bounded from below by 0, whereas $J(S_k'^-)$ is bounded from above by $J(S_k'^{ub})$, where $S_k'^{ub}$ is any $(K-1)$-dimensional hyper-rectangle that contains $S_k'^-$, for instance, the hyper-rectangle $\pi_{\mathcal{K}\setminus\{k\}}(\Pi_{j=1}^K\{\alpha_j, \ldots, n_{j,\max}^{\mathrm{FR}}\})$. Hence, with this choice of $S_k'^{ub}$, one gets

$$\begin{aligned} J\left(S_k'^{ub}\right) &= \sum_{j\in\mathcal{K}\setminus\{k\}} r_j x_j\left(\alpha_j, n_{j,\max}^{\mathrm{FR}}\right) \\ &\geq r_k\left(\alpha_k - x_k\left((\mathbf{C}-\mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1\right)\right). \end{aligned}$$

Since $\lambda_1(\cdot), \lambda_2(\cdot), \ldots, \lambda_K(\cdot)$ are nonincreasing, by Lemma VII.5, we get

$$\begin{aligned} &\sum_{j\in\mathcal{K}\setminus\{k\}} r_j x_j\left(0, n_{j,\max}^{\mathrm{FR}} - \alpha_j\right) \\ &\qquad\geq r_k\left(\alpha_k - x_k\left((\mathbf{C}-\mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k} - 1\right)\right) \end{aligned}$$

whereas

$$\sum_{j\in\mathcal{K}\setminus\{k\}} r_j x_j\left(0, n_{j,\max}^{\mathrm{FR}}\right) \geq \sum_{j\in\mathcal{K}\setminus\{k\}} r_j x_j\left(0, n_{j,\max}^{\mathrm{FR}} - \alpha_j\right)$$

follows by the expression of the function $x_j(\cdot,\cdot)$ in (9). ■

Proof of Theorem IV.8: By Proposition IV.7, any corner point $\boldsymbol{\alpha}$ of Ω^{o} belongs to the grid G, so it is associated to a vector $\mathbf{C}_G$, where $\alpha_h = \mathbf{C}_{G,h}$ for each h.

(i) If $\alpha_k \neq 0$, then by Lemma VII.9, the condition (32) holds. However, this is in contradiction with the assumption (10), so one obtains $\alpha_k = 0$.

(ii) Proceeding likewise in the proof of (i), it follows that all the potential corner points of Ω^{o} have $\alpha_k = 0$, for all $k \in \mathcal{K}$. Since (0, 0, …, 0) is never a corner point, it follows that Ω^{o} has no corner points, so it is the complete sharing policy.

■

Sketch of the Proof of Proposition III.2 for $K > 2$: The main difference with respect to the case $K = 2$ is in the construction of the IR_Ω set S^- and the IA_Ω set S^+ in Step a) of the proof. In particular, two sets S^- and S^+ satisfying a straightforward extension of Proposition VII.2 to $K > 2$ can be built in a similar way as the sets S_k^- and S_k^+ in (33) and (34). Regarding Step b), the condition $|I(\Omega)| = 2$ is replaced by $|I(\Omega)| = K$. ■

Sketch of the Proof of Proposition III.5 for $K > 2$: The proof is based on an induction argument (the result has been already shown to hold for two classes). Suppose that the statement is true for any $(K-1)$-dimensional CC set. Now, for a given K-dimensional CC set $\Omega \subseteq \Omega_{\mathrm{FR}}$ and for $j = 0,\ldots,n_{K,\max}^{\mathrm{FR}}$, we consider the projections $\Omega_j^K := \pi_{\mathcal{K}\setminus\{K\}}(\Omega \cap \mathcal{P}_K(j))$ and $\Omega_{\mathrm{FR,j}}^K := \pi_{\mathcal{K}\setminus\{K\}}(\Omega_{\mathrm{FR}} \cap \mathcal{P}_K(j))$. Since all their elements $\mathbf{n}$ have $n_K = 0$, the sets Ω_j^K and $\Omega_{\mathrm{FR,j}}^K$ are actually $(K-1)$-dimensional CC sets, so, by the induction hypothesis, one has the representation $\Omega_j^K = (\Omega_{\mathrm{FR,j}}^K \setminus \cup_{l\in I(\Omega_j^K)} C_{l,j}^{K,-})$, where the sets $C_{l,j}^{K,-}$ are $(K-1)$-dimensional hyperoctants. Then, by increasing the index j from $j=0$ to $j = n_{K,\max}^{\mathrm{FR}}$, one can show that each time new corner points of Ω_K^j show up, they generate corner points of Ω with coordinates equal to those of the new corner points of Ω_j^K, with the exception of the Kth coordinate, which is equal to j. Finally, sets C_i^- of the required form (i.e., K-dimensional hyperoctants associated to the just determined corner points of Ω) can be readily generated from the ones $C_{l,j}^{K,-}$ by adding a suitable constraint on the Kth coordinate. ■

Proof of Proposition VI.1: By the definition of the function $x_k(\cdot,\cdot)$, we get

$$x_1(0,B_1) = \frac{\sum_{j=0}^{B_1} j\frac{\rho_1^j}{j!}}{\sum_{j=0}^{B_1}\frac{\rho_1^j}{j!}} = \frac{\rho_1\sum_{j=1}^{B_1}\frac{\rho_1^{j-1}}{(j-1)!}}{\sum_{j=0}^{B_1}\frac{\rho_1^j}{j!}} = \frac{\rho_1\sum_{j=1}^{B_1-1}\frac{\rho_1^j}{j!}}{\sum_{j=0}^{B_1}\frac{\rho_1^j}{j!}}. \tag{35}$$

As $\sum_{j=1}^{B_1-1}(\rho_1^j/j!)/\sum_{j=0}^{B_1}(\rho_1^j/j!)$ is smaller than 1, one has

$$x_1(0,B_1) < \rho_1.$$

The following result is obtained in the same way:

$$x_2(0,B_2) < \rho_2.$$

Finally, one has

$$\frac{1}{L_1} \leq x_1(0,B_1) < \rho_1 < R < \frac{1}{\rho_2} < \frac{1}{x_2(0,B_2)} \leq L_2.$$

■

Proof of Proposition VI.2: Since $\mathbf{C}_{G,k} - x_k((\mathbf{C}-\mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k}-1) \geq 1$ for $\mathbf{C}_{G,k} \neq 0$, we have

$$\frac{\sum_{j\in\mathcal{K}\setminus\{k\}} r_j x_j\left(0, n_{j,\max}^{\mathrm{FR}}\right)}{\mathbf{C}_{G,k} - x_k\left((\mathbf{C}-\mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k}-1\right)} \leq \sum_{j\in\mathcal{K}\setminus\{k\}} r_j x_j\left(0, n_{j,\max}^{\mathrm{FR}}\right).$$

Moreover, proceeding likewise in formula (35), one obtains

$$\sum_{j\in\mathcal{K}\setminus\{k\}} r_j x_j\left(0, n_{j,\max}^{\mathrm{FR}}\right) < \sum_{j\in\mathcal{K}\setminus\{k\}} r_j\rho_j = \sum_{j\in\mathcal{K}\setminus\{k\}} r_j\frac{\lambda_j}{\mu_j}$$

which concludes the proof. ■

Proof of Proposition VI.3: The proposition is an immediate consequence of Lemma VII.9 and the characterization of the restricted complete sharing policy provided in Section VI.[8] ■

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$$\mathbf{C}_{G,k} - x_k\left((\mathbf{C}-\mathbf{e}_k)_{G,k}, \mathbf{C}_{G,k}-1\right) > 1 \tag{36}$$

for nondegenerate choices of $\lambda_k(\cdot)$ and μ_k and of the feasibility region.

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